Proposal: Practical Groth16 Aggregation

No Author Given

March 4, 2021

Abstract

Groth16 arguments of knowledge have become a de-facto standard used in several blockchain projects due to the constant size of their proofs and their appealing verifier time. In this context, all nodes verify proofs posted on each block individually. Given an upper bound on the number of proofs allowed per blocks this solution doesn't scale further. [BMM+19] presents a way to create an aggregate proof from n Groth16 proofs with a $O(logn)$ size and verifier time. However, the protocol requires a custom trusted setup, different than the Groth16 one. In this work, we modify the previous construction with a different commitment scheme that allows to aggregate existing Groth16 proofs by re-using existing powers of tau ceremonies transcript. Our protocol can aggregate 1024 proofs in 2s and verifies them in 23ms, yielding an exponentially faster verification mechanism than batching.

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1 Introduction

1.1 Arguments of Knowledge

In decentralised systems, there is need for protocols that enable a prover to post a statement together with a *short* proof, such that any verifier can publicly check that the statement (e.g., correctness of a computation, claims of storage etc.) is true with fewer resources, e.g., faster than re-executing the function.

The two key properties in these practical settings are the size of the proof and the verification time. A popular application is a blockchain where nodes verify all proofs posted in each blocks and thus requiring a low verification time.

However, when it comes to optimization of communication complexity in proof systems, it has been shown that statistically-sound proofs are unlikely to allow for significant improvements in proof size. [Wee05] shows that when considering proof systems for NP language, unless some complexity-theoretic collapses occur, statistical soundness requires the prover to communicate, roughly, as much information as the size of the witness.

The search for ways to beat this bound motivated the study of *computationally-sound* proof systems, also called *argument systems*, where soundness is required to hold only against *computationally bounded* provers. When restricting ourselves to *computational soundness*, we can build argument systems with proof sizes smaller than the length of the witness.

For the other aspect, what we call in the literature *succinct verification*, requires for a verifier to check a nondeterministic polynomial-time computation in time that is much shorter than the time required to run the computation given the NP witness.

Succinct Non-interactive ARguments of Knowledge (SNARKs) are proof systems that fulfill these requirements and they are more and more popular in real-world applications. There has been a series of works on constructing SNARKs [Gro10, Lip12, BCI$^+$13, GGPR13, PHGR13, Lip13, BCTV14, Gro16] with constant-size proofs.

1.2 Motivation

While there are many SNARKs with constant-sized proofs and succinct verifiers, in a blockchain settings, all nodes in the network need to verify many proofs coming from different provers individually. This situations can create a bottleneck where the bandwitdh is limited by the number of proofs the chain can verify in an epoch, i.e. the maximum number of proofs that can be included in a block. One concrete example is the proof of space Filecoin [Lab18] blockchain where miners must post a Groth16 proof that they correctly computed proof of space [Fis19] to emboard new storage on the network. Each proof guarantees that the miner correctly "reserved" 32GB of space to store specific files. However, the chain is seeing a large amount of proofs posted each day to emboard new storage: 15 PiB in average per day which is approximatively 500k Groth16 proofs posted each day. This scalability issues requires a new way for miners to prove their newly emboarded storage. [BMM$^+$19] presents a framework for aggregating Groth16 proofs into a $O(logn)$ sized final proof with a verification running in $O(logn)$ ($O(n)$ for the public inputs). This construction can be applied to aggregating Groth16 proofs but requires a specific trusted setup to construct the structured reference string necessary to verify such aggregated proofs. Given the delicate trust assumptions around trusted setup, we explored ways to aggregate Groth16 proofs by limit-

ing ourselves in using the already available Groth16 trusted setup transcript and therefore avoiding an additional trust assumptions for existing systems.

1.3 Contribution

Our construction is based on techniques from [BMM+19] and follows exactly the same framework, but aims at making this realizable in practice without the need of further trusted setup ceremonies and taking full advantage of existing building blocks for further optimisations.

Our focus is to realise aggregation for specifically Groth16 proofs [Gro16] all generated using the same SRS, as opposed to the generic result in [BMM+19] that aims to apply to any kind of pairing-based SNARKs schemes and also allow to aggregate proofs generated using different SRS.

The Argument for Aggregation that we propose is based only on black-box composition of standard primitives and uses existing setup ceremonies, more specifically the exact one that generates the structured reference string of Groth16 itself.

The produced aggregated proof has logarithmic size and verifier runs in logarithmic time in the number of proofs to be aggregated. The reference string needed to compute and verify this aggregated proof can be constructed from any pairs of powers of tau ceremonies (for example the Zcash [zca18] and Filecoin [Fil20]).

We present an argument for aggregating n Groth16 SNARKs into an $O(\log n)$ sized proof. Our protocol is exponentially more efficient than aggregating these SNARKs via batching: it takes 31ms to verify an aggregated proof for 8192 proofs versus 491ms when doing batch verification and can aggregate 1024 proofs in 2s.

2 Preliminaries

2.1 Notations and General Background

Bilinear Groups. A bilinear group is given by a description $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T)$ such that

- p is prime, so $\mathbb{Z}_p^* = \mathbb{F}$ is a field.
- $\mathbb{G}_1 = \langle g \rangle, \mathbb{G}_2 = \langle h \rangle$ are multiplicative cyclic groups of prime order p.
- $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ is a bilinear asymmetric map (pairing), which means that $\forall a, b \in \mathbb{Z}_p : e(g^a, h^b) = e(g, h)^{ab}$.
- Then we implicitly have that $e(g, h)$ generates $\mathbb{G}_T$.
- Membership in $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ can be efficiently decided, group operations and the pairing $e(\cdot, \cdot)$ are efficiently computable, generators can efficiently be sampled, and the descriptions of the groups and group elements each have linear size.

Vectors. For n-dimensional vectors $\mathbf{a} \in \mathbb{Z}_p^n, \mathbf{A} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n$, we denote their i-th entry by $a_i \in \mathbb{Z}_p, A_i \in \mathbb{G}_1, B_i \in \mathbb{G}_2$ respectively.

Let $\mathbf{A} \| \mathbf{A}' = (A_0, \ldots, A_{n-1}, A'_0, \ldots, A'_{n-1})$ be the concatenation of two vectors $\mathbf{A}, \mathbf{A}' \in \mathbb{G}_1^n$.

We write $\mathbf{A}_{[:\ell]} = (A_0, \ldots, A_{\ell-1}) \in \mathbb{G}_1^\ell$ and $\mathbf{A}_{[\ell:]} = (A_\ell, \ldots, A_{n-1}) \in \mathbb{G}_1^{n-\ell}$ to denote slices of vectors $\mathbf{A} \in \mathbb{G}_1^n$ for $0 \leq \ell < n - 1$.

Same notation holds for vectors $\mathbf{B} \in \mathbb{G}_2^n$ in the second source group.

Inner pairing product. We write group operations as multiplications.

We define $\mathbf{A}^x = (A_0^x, \ldots, A_{n-1}^x) \in \mathbb{G}_1^n$ for a scalar $x \in \mathbb{Z}_p$ and a vector $\mathbf{A} \in \mathbb{G}_1^n$.

We define $\mathbf{A}^{\mathbf{x}} = (A_0^{x_0}, \ldots, A_{n-1}^{x_{n-1}}) \in \mathbb{G}_1^n$ for vectors $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{Z}_p^n, \mathbf{A} \in \mathbb{G}_1^n$..

We define $\mathbf{A} * \mathbf{B} \coloneqq \prod_{i=0}^{n} e(A_i, B_i)$ for a pair of source group vectors $\mathbf{A} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n$.

We define $\mathbf{A} \circ \mathbf{A'} \coloneqq (A_0 A'_0, \ldots, A_{n-1} A'_{n-1})$ for two vectors in the same group $\mathbf{A}, \mathbf{A'} \in \mathbb{G}_1^n$.

Relations. We use the notation $\mathcal{R}$ to denote an efficiently decidable binary relation.

For pairs $(u, w) \in \mathcal{R}$ we call u the statement and w the witness. We write $\mathcal{R} = \{(u; w) : p(u, w)\}$ to describe a NP relation $\mathcal{R} \subseteq \{0,1\}^* \times \{0,1\}^*$ between instances u and witnesses w decided by the polynomial-time predicate $p(\cdot, \cdot)$. Let $L_{\mathcal{R}}$ be the language consisting of statements u for which there exist matching witnesses in $\mathcal{R}$.

Polynomial-Time Algorithms. Unless otherwise specified, all the algorithms defined throughout this work are assumed to be probabilistic Turing machines with running time bounded by a polynomial in his input size, where the expectation is taken over the random coins of the algorithm - i.e., PPT.

If $\mathcal{A}$ is a randomized algorithm, we use $y \leftarrow\!\$\, \mathcal{A}(x)$ to denote that y is the output of $\mathcal{A}$ on x. We write $x \leftarrow\!\$\, X$ to mean sampling a value x uniformly from the set X.

By writing $\mathcal{A} \| \chi_{\mathcal{A}}(\sigma)$ we denote the execution of $\mathcal{A}$ followed by the execution of $\chi_{\mathcal{A}}$ on the same input σ and with the same random coins. The output of the two are separated by a semicolon.

Security Parameter. We denote the computational security parameter with $\lambda \in \mathbb{N}$: A cryptosystem provides λ bits of security if it requires 2^λ elementary operations to be broken.

We say that a function is *negligible* in λ, and we denote it by $\mathsf{negl}(\lambda)$, if it is a $f(\lambda) = \mathcal{O}(\lambda^{-c})$ for any fixed constant c.

Adversaries. Adversaries are PPT algorithms denoted with calligraphic letters ($e.g. \mathcal{A}, \mathcal{B}$). They will be usually be modeled as efficient algorithms taking 1^λ as input.
We define the adversary's advantage as a function of parameters to be $\Pr[\mathcal{A} \text{ wins}]$. For a system to be secure, we require that for any efficient adversary $\mathcal{A}$, the advantage of $\mathcal{A}$ is negligible in the security parameter.

Common and Structured Reference String. The common reference string (CRS) model, introduced by Damgård [Dam00], captures the assumption that a trusted setup in which all involved parties get access to the same string crs taken from some distribution exists. Schemes proven secure in the CRS model are secure given that the setup was performed correctly. The common reference string model is a generalization of the common random string model, in which is the uniform distribution of bit strings. We will use the recommended terminology "Structured Reference String" (SRS) since all our crs are structured.

Generic Group Model. The generic group model [Sho97, Mau05] is an idealised cryptographic model, where algorithms do not exploit any special structure of the representation of the group elements and can thus be applied in any cyclic group.

In this model, the adversary is only given access to a randomly chosen encoding of a group, instead of efficient encodings, such as those used by the finite field or elliptic curve groups used in practice. The model includes an oracle that executes the group operation. This oracle takes two encodings of group elements as input and outputs an encoding of a third element. If the group should allow for a pairing operation this operation would be modeled as an additional oracle.

One of the primary uses of the generic group model is to analyse computational hardness assumptions. An analysis in the generic group model can answer the question: "What is the fastest generic algorithm for breaking a cryptographic hardness assumption?". A generic algorithm is an algorithm that only makes use of the group operation, and does not consider the encoding of the group.

2.2 Cryptographic Primitives

2.2.1 SNARKs

Let $\mathcal{R}$ be an efficiently computable binary relation which consists of pairs of the form (u, w) and let $L_{\mathcal{R}}$ be the language associated with $\mathcal{R}$.

A Proof or Argument System for $\mathcal{R}$ consists in a triple of PPT algorithms $\Pi = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ defined as follows:

$\mathsf{Setup}(1^\lambda, \mathcal{R}) \to \mathsf{crs}$**:** takes a security parameter λ and a binary relation $\mathcal{R}$ and outputs a common (structured) reference string crs.

$\mathsf{Prove}(\mathsf{crs}, u, w) \to \pi$**:** on input crs, a statement u and the witness w, it outputs an argument π.

$\mathsf{Verify}(\mathsf{crs}, u, \pi) \to 1/0$**:** on input crs, a statement u, and a proof π, it outputs either 1 indicating accepting the argument or 0 for rejecting it.

We call Π a Succinct Non-interactive ARgument of Knowledge (SNARK) if further it is complete, succinct and satisfies *Knowledge Soundness* (also called *Proof of Knowledge*).

Non-black-box Extraction. The notion of *Knowledge Soundness* requires the existence of an extractor that can compute a witness whenever the prover $\mathcal{A}$ produces a valid argument. The extractor we defined bellow is non-black-box and gets full access to the prover's state, including any random coins. More formally, a SNARK satisfies the following definition:

Definition 2.1 (SNARK)**.** $\Pi = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *is a SNARK for an NP language* $L_{\mathcal{R}}$ *with corresponding relation* $\mathcal{R}$*, if the following properties are satisfied.*

Completeness. *For all* $(x, w) \in \mathcal{R}$*, the following holds:*

$$\Pr\left(\mathsf{Verify}(\mathsf{crs}, u, \pi) = 1 \,\middle|\, \begin{matrix} \mathsf{crs} \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{R}) \\ \pi \leftarrow \mathsf{Prove}(\mathsf{crs}, u, w) \end{matrix}\right) = 1$$

Knowledge Soundness. *For any* PPT *adversary* $\mathcal{A}$*, there exists a* PPT *extractor* $\mathsf{Ext}_{\mathcal{A}}$ *such that the following probability is negligible in* λ*:*

$$\Pr\left(\begin{matrix}\mathsf{Verify}(\mathsf{crs},u,\pi)=1\\ \wedge\,\mathcal{R}(u,w)=0\end{matrix}\middle|\begin{matrix}\mathsf{crs}\leftarrow\mathsf{Setup}(1^\lambda,\mathcal{R})\\ ((u,\pi);w)\leftarrow\mathcal{A}\|\chi_{\mathcal{A}}(\mathsf{crs})\end{matrix}\right)=\mathsf{negl}(\lambda).$$

Succinctness. *For any u and w, the length of the proof π is given by $|\pi| = \mathsf{poly}(\lambda) \cdot \mathsf{polylog}(|u| + |w|)$.*

Zero-Knowledge. A SNARK is zero-knowledge if it does not leak any information besides the truth of the statement. More formally:

Definition 2.2 (zk-SNARK)**.** *A SNARK for a relation $\mathcal{R}$ is a zk-SNARK if there exists a* PPT *simulator $(\mathcal{S}_1, \mathcal{S}_2)$ such that $\mathcal{S}_1$ outputs a simulated common reference string* crs *and trapdoor* td*; $\mathcal{S}_2$ takes as input* crs*, a statement u and* td*, and outputs a simulated proof π; and, for all* PPT *(stateful) adversaries $(\mathcal{A}_1, \mathcal{A}_2)$, for a state* st*, the following is negligible in λ:*

$$\left|\Pr\left(\begin{matrix}(u,w)\in\mathcal{R}\wedge\\ \mathcal{A}_2(\pi,\mathsf{st})=1\end{matrix}\middle|\begin{matrix}\mathsf{crs}\leftarrow\mathsf{Setup}(1^\lambda)\\ (u,w,\mathsf{st})\leftarrow\mathcal{A}_1(1^\lambda,\mathsf{crs})\\ \pi\leftarrow\mathsf{Prove}(\mathsf{crs},u,w)\end{matrix}\right)-\Pr\left(\begin{matrix}(u,w)\in\mathcal{R}\wedge\\ \mathcal{A}_2(\pi,\mathsf{st})=1\end{matrix}\middle|\begin{matrix}(\mathsf{crs},\mathsf{td})\leftarrow\mathcal{S}_1(1^\lambda)\\ (u,w,\mathsf{st})\leftarrow\mathcal{A}_1(1^\lambda,\mathsf{crs})\\ \pi\leftarrow\mathcal{S}_2(\mathsf{crs},\mathsf{td},u)\end{matrix}\right)\right|=\mathsf{negl}(\lambda).$$

2.2.2 Commitment Schemes

A non-interactive commitment scheme allows a sender to create a commitment to a secret value. It may later open the commitment and reveal the value or some information about the value in a verifiable manner. More formally:

Definition 2.3 (Non-Interactive Commitment)**.** *A non-interactive commitment scheme is a pair of algorithms* $\mathsf{Com} = (\mathsf{KG}, \mathsf{CM})$*:*

$\mathsf{KG}(1^\lambda) \to \mathsf{ck}$**:** *given a security parameter λ, it generates a commitment public key* ck*. This* ck *implicitly specifies a message space M_{ck}, a commitment space C_{ck} and (optionally) a randomness space R_{ck},. This algorithm is run by a trusted or distributed authority.*

$\mathsf{CM}(\mathsf{ck}; m) \to C$**:** *given* ck *and a message m, outputs a commitment C. This algorithm specifies a function $\mathsf{Com}_{\mathsf{ck}} : M_{\mathsf{ck}} \times \mathcal{R}_{\mathsf{ck}} \to C_{\mathsf{ck}}$. Given a message $m \in M_{\mathsf{ck}}$, the sender (optionally) picks a randomness $\rho \in R_{\mathsf{ck}}$ and computes the commitment $C = \mathsf{Com}_{\mathsf{ck}}(m, \rho)$*

For deterministic commitments we simply use the notation $C = \mathsf{CM}(\mathsf{ck}; m) \coloneqq \mathsf{Com}_{\mathsf{ck}}(m, 0)$, while for randomised ones we write $C \leftarrow\!\!\$\, \mathsf{CM}(\mathsf{ck}; m) \coloneqq \mathsf{Com}_{\mathsf{ck}}(m, \rho)$.

A commitment scheme is asked to satisfy one or more of the following properties:

Binding Definition. It is computationally hard, for any PPT adversary $\mathcal{A}$, to come up with two different openings $m \neq m^* \in M_{\mathsf{ck}}$ for the same commitment C. More formally:

Definition 2.4 (Computationally Binding Commitment)**.** *A commitment scheme* $\mathsf{Com} = (\mathsf{KG}, \mathsf{CM})$ *is computationally binding if for any* PPT *adversary* $\mathcal{A}$*, the following probability is negligible*

$$\Pr\left[\begin{array}{c|r} m \neq m^* & \mathsf{ck} \leftarrow \mathsf{KG}(1^\lambda) \\ \wedge\ \mathsf{CM}(\mathsf{ck}; m) = \mathsf{CM}(\mathsf{ck}; m^*) = C & (C; m, m^*) \leftarrow \mathcal{A}(\mathsf{ck}) \end{array}\right] = \mathsf{negl}(\lambda).$$

Hiding Definition. A commitment can be hiding in the sense that it does not reveal the secret value that was committed.

Definition 2.5 (Statistically Hiding Commitment)**.** *A commitment scheme* $\mathsf{Com} = (\mathsf{KG}, \mathsf{CM})$ *is statistically hiding if it is statistically hard, for any* PPT *adversary* $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$*, to first generate two messages* $\mathcal{A}_0(\mathsf{ck}) \to m_0, m_1 \in M_{\mathsf{ck}}$ *such that* $\mathcal{A}_1$ *can distinguish between their corresponding commitments* C_0 *and* C_1 *where* $C_0 \leftarrow\!\$\, \mathsf{CM}(\mathsf{ck}; m_0)$ *and* $C_1 \leftarrow\!\$\, \mathsf{CM}(\mathsf{ck}; m_1)$*.*

$$\Pr\left[b = b' \;\middle|\; \begin{array}{r} \mathsf{ck} \leftarrow \mathsf{KG}(1^\lambda) \\ (m_0, m_1) \leftarrow \mathcal{A}_0(\mathsf{ck}) \\ b \leftarrow \{0,1\},\ C_b \leftarrow\!\$\, \mathsf{CM}(\mathsf{ck}; m_b) \\ b' \leftarrow \mathcal{A}_1(\mathsf{ck}, C_b) \end{array}\right] = \mathsf{negl}(\lambda).$$

Homomorphic Commitment Scheme. A commitment scheme can also be homomorphic, either in the space of messages or in the space of keys or in both. We call the later *doubly-homomorphic* commitments.

- *Message Homomorphism.* For a group law $+$ on the message space M_{ck} and $\oplus$ on the commitment space C_{ck}, we have that from $C_0 = \mathsf{CM}(\mathsf{ck}; m_0)$ and $C_1 = \mathsf{CM}(\mathsf{ck}; m_1)$, one can efficiently generate $C = \mathsf{CM}(\mathsf{ck}; m_0 + m_1)$ by computing $C = C_0 \oplus C_1 = \mathsf{CM}(\mathsf{ck}; m_0 + m_1)$.
- *Key Homomorphism.* For a group law $\star$ on the key space K_{ck}, and $\oplus$ on the commitment space C_{ck}, we have that from $C_0 = \mathsf{CM}(\mathsf{ck}_0; m)$ and $C_1 = \mathsf{CM}(\mathsf{ck}_1; m)$, one can efficiently generate C so that $C = C_0 \oplus C_1 = \mathsf{CM}(\mathsf{ck}_0 \star \mathsf{ck}_1; m)$.

2.2.3 Polynomial Commitments

Polynomial commitments (PCs) first introduced by [KZG10] are commitments for the message space $\mathbb{F}^{\leq d}[X]$, the ring of polynomials in X with maximum degree $d \in \mathbb{N}$ and coefficients in the field $\mathbb{F} = \mathbb{Z}_p$, that support an interactive argument of knowledge $(\mathsf{KG}, \mathsf{Open}, \mathsf{Check})$ for proving the correct evaluation of a committed polynomial at a given point without revealing any other information about the committed polynomial.

A polynomial commitment scheme over a field family $\mathcal{F}$ consists in 4 algorithms $\mathsf{PC} = (\mathsf{KG}, \mathsf{CM}, \mathsf{Open}, \mathsf{Check})$ defined as follows:

$\mathsf{KG}(1^\lambda, d) \to (\mathsf{ck}, \mathsf{vk})$**:** given a security parameter λ fixing a field $\mathcal{F}_\lambda$ family and a maximal degree d samples a group description gk containing a description of a field $\mathbb{F} \in \mathcal{F}_\lambda$, and commitment and verification keys $(\mathsf{ck}, \mathsf{vk})$. We implicitly assume ck and vk each contain gk.

$\mathsf{CM}(\mathsf{ck}; f(X); r) \to C$**:** given ck and a polynomial $f(X) \in \mathbb{F}^{\leq d}[X]$ outputs a commitment C.

$\mathsf{Open}(\mathsf{ck}; C, x, y; f(X), r) \to \pi$**:** given a commitment C, an evaluation point x, a value y, the polynomial $f(X) \in \mathbb{F}[X]$ and a randomness r, it output a prove π for the relation:

$$\mathcal{R}_{\mathsf{KZG}} := \left\{ (\mathsf{ck}, C, x, y; f(X), r) \ : \begin{array}{r} C = \mathsf{CM}(\mathsf{ck}; f(X); r) \\ \wedge \ \deg(f(X)) \leq d \\ \wedge \ y = f(x) \end{array} \right\}$$

$\mathsf{Check}(\mathsf{vk}, C, x, y, \pi) \to 1/0$: Outputs 1 if the proof π verifies and 0 if π is not a valid proof for the opening (C, x, y).

A polynomial commitment satisfy hiding property and an extractable version of binding stated as follows:

Definition 2.6 (Computational Knowledge Binding)**.** *For every* PPT *adversary* $\mathcal{A}$ *that produces a valid commitment-proof* C, π *that verifies, i.e. such that* $\mathsf{Check}(\mathsf{vk}, C, x, y, \pi) = 1$*, there is an extractor* $\mathsf{Ext}_{\mathcal{A}}$ *that is able to output a pre-image polynomial* $f(X)$ *with overwhelming probability:*

$$\Pr\left[C = \mathsf{CM}(\mathsf{ck}; f(X); r) \;\middle|\; \begin{array}{r} \mathsf{ck} \leftarrow \mathsf{KG}(1^\lambda, d) \\ (C, \pi; f(X)) \leftarrow (\mathcal{A} \| \mathsf{Ext}_{\mathcal{A}})(\mathsf{ck}) \\ \mathsf{Check}(\mathsf{vk}, C, x, y, \pi) = 1 \end{array} \right] = 1 - \mathsf{negl}(\lambda).$$

2.3 Assumptions

2.3.1 ASSGP - Auxiliary Structured Single Group Pairing

Informally this assumption can be stated as: We assume that a PPT adversary cannot find a vector of group elements $A_1, \ldots, A_q \in \mathbb{G}_1$ such as:

1. $\exists A_i \neq 1$
2. $e(A_1, h^a) \ldots . e(A_q, h^{a^q}) = 1$
3. $e(A_1, h^b) \ldots . e(A_q, h^{b^q}) = 1$

Formally, q-Auxiliary Structured Single Group Pairing (q-ASSGP) assumption can be stated as:

Assumption 2.7 (ASSGP)**.** *The* q*-Auxiliary Structured Single Group Pairing (*q*-ASSGP) assumption holds for the bilinear group generator* $\mathcal{G}$ *if for all* PPT *adversaries* $\mathcal{A}$ *we have, on the probability space* $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T) \leftarrow \mathcal{G}(1^\lambda)$*,* $g \leftarrow\$ \mathbb{G}_1, h \leftarrow\$ \mathbb{G}_2$ *and* $a, b \leftarrow\$ \mathbb{Z}_p$ *the following holds:*

$$\Pr\left[\begin{array}{c} (A_0, \ldots, A_{q-1}) \neq \mathbf{1}_{\mathbb{G}_1} \\ \wedge \ \prod_{i=0}^{q-1} e(A_i, h^{a^i}) = 1_{\mathbb{G}_T} \\ \wedge \ \prod_{i=0}^{q-1} e(A_i, h^{b^i}) = 1_{\mathbb{G}_T} \end{array} \;\middle|\; \begin{array}{r} g \leftarrow\$ \mathbb{G}_1, h \leftarrow\$ \mathbb{G}_2, a, b \leftarrow\$ \mathbb{Z}_p \\ \sigma \leftarrow ([g^{b^i}]_{i=0}^{2q-1}, [g^{a^i}]_{i=0}^{2q-1}, [h^{b^i}]_{i=0}^{2q-1}, [h^{a^i}]_{i=0}^{2q-1}) \\ (A_1, \ldots, A_q) \leftarrow \mathcal{A}(\mathsf{gk}, \sigma) \end{array} \right] = \mathsf{negl}(\lambda)$$

We refer to this assumption defined for single group paring with elements from second group $\mathbb{G}_2$ as the q-ASSGP2 assumption and also define its dual, the q-ASSGP1 assumption, by swapping $\mathbb{G}_1$ and $\mathbb{G}_2$ in the definition above.

Lemma 2.8. *The* q*-ASSGP assumption holds in the generic group model.*

The proof of the lemma can be find in Appendix A.1.

2.3.2 ASDGP - Auxiliary Structured Double Group Pairing

Informally we assume that a PPT adversary can not find a pair of vectors $(A_1, \ldots, A_n) \in \mathbb{G}_1$ and $(B_1, \ldots, B_n) \in \mathbb{G}_2$ such that:

1. $\exists A_i \neq 1$ or $B_i \neq 1$
2. $e(A_1, h^a) \cdot \ldots e(A_n, h^{a^n}) \cdot e(g^{a^{n+1}}, B_1) \ldots e(g^{a^{2n}}, B_n) = 1$
3. $e(A_1, h^b) \cdot \ldots e(A_n, h^{b^n}) \cdot e(g^{b^{n+1}}, B_1) \cdot \ldots e(g^{b^{2n}}, B_n) = 1$

Formally, q-Auxiliary Structured Double Group Pairing (q-ASDGP) assumption can be stated as:

Assumption 2.9 (ASDGP)**.** *The q-ASDGP assumption holds for the bilinear group generator $\mathcal{G}$ if for all* PPT *adversaries $\mathcal{A}$ we have, on the probability space $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T) \leftarrow \mathcal{G}(1^\lambda)$, $g \leftarrow\!\$\, \mathbb{G}_1, h \leftarrow\!\$\, \mathbb{G}_2$ and $a, b \leftarrow\!\$\, \mathbb{Z}_p$ the following holds:*

$$\Pr\left[\begin{array}{c|r} (\mathbf{A} \neq \mathbf{1}_{\mathbb{G}_1} \vee \mathbf{B} \neq \mathbf{1}_{\mathbb{G}_2}) & g \leftarrow\!\$\, \mathbb{G}_1, h \leftarrow\!\$\, \mathbb{G}_2, a, b \leftarrow\!\$\, \mathbb{Z}_p \\ \wedge \prod_{i=0}^{q-1} e(A_i, h^{a^i}) \prod_{i=q}^{2q-1} e(g^{a^i}, B_i) = 1_{\mathbb{G}_T} & \sigma \leftarrow ([g^{a^i}, g^{b^i}, h^{a^i}, h^{b^i}]_{i=0}^{2q-1}) \\ \wedge \prod_{i=0}^{q-1} e(A_i, h^{b^i}) \prod_{i=q}^{2q-1} e(g^{b^i}, B_i) = 1_{\mathbb{G}_T} & (\mathbf{A}, \mathbf{B}) \leftarrow \mathcal{A}(\mathsf{gk}, \sigma) \end{array}\right] = \mathsf{negl}(\lambda)$$

Lemma 2.10. *The q-ASDGP assumption holds in the generic group model.*

The proof of the lemma can be found in Appendix A.2.

3 Groth16 Aggregation

In this section we present the framework for aggregating multiple Groth16 proofs for the same SRS (same verification key). This was first introduced by the work [BMM+19].

3.1 Background on Groth16

Before presenting the aggregation argument scheme, we recall here [Gro16] SNARK scheme construction.

Let C be an arithmetic circuit over $\mathbb{Z}_p$, with m wires and d multiplication gates.

Let $Q = (t(x), \{v_k(x), w_k(x), y_k(x)\}_{k=0}^m)$ be a QAP which computes C.

We denote by $I_{io} = \{1, 2, \ldots \ell\}$ the indices corresponding to the public input and public output values of the circuit wires and by $I_{mid} = \{\ell + 1, \ldots m\}$, the wire indices corresponding to non-input, non-output intermediate values (for the witness).

We describe SNARK= $(\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ scheme in [Gro16] that consists in 3 algorithms as per Figure 1.

Note that the Groth16 SRS consist in consecutive powers of some random evaluation point s in both groups $\mathbb{G}_1$ and $\mathbb{G}_2$:

$$\{g^{s^i}\}_{i=0}^{d-1} \in \mathbb{G}_1^d, \quad \{h^{s^i}\}_{i=0}^{d-1} \in \mathbb{G}_2^d.$$

and some additional polynomials evaluated in this random point s.

$\underline{\text{Groth.}\mathsf{Setup}(1^\lambda, \mathcal{R}) :}$

$\alpha, \beta, \gamma, \delta \leftarrow\$ \mathbb{Z}_p^*, \quad s \leftarrow\$ \mathbb{Z}_p^*,$

$$\mathsf{crs} = \left(\mathsf{QAP}, g^\alpha, g^\beta, g^\delta, \{g^{s^i}\}_{i=0}^{d-1}, \ \left\{ g^{\frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\gamma}} \right\}_{k=0}^{\ell}, \left\{ g^{\frac{\beta v_k(s) + alpha w_k(s) + y_k(s)}{\delta}} \right\}_{k>\ell}, \right.$$
$$\left. \left\{ g^{\frac{s^i t(s)}{\delta}} \right\}_{i=0}^{d-2}, h^\beta, h^\gamma, h^\delta, \{h^{s^i}\}_{i=0}^{d-1} \right)$$

$$\mathsf{vk} := \left(P = g^\alpha, Q = h^\beta, \ \left\{ S_k = g^{\frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\gamma}} \right\}_{k=0}^{\ell}, H = h^\gamma, D = h^\delta \right)$$

$\mathsf{td} = (s, \alpha, \beta, \gamma, \delta)$

return $(\mathsf{crs}, \mathsf{td})$

$\underline{\text{Groth.}\mathsf{Prove}(\mathsf{crs}, u, w)}$

$u = (a_1, \ldots, a_\ell), \ a_0 = 1$

$w = (a_{\ell+1}, \ldots, a_m)$

$v(x) = \sum_{k=0}^{m} a_k v_k(x)$

$v_{mid}(x) = \sum_{k \in I_{mid}} a_k v_k(x)$

$w(x) = \sum_{k=0}^{m} a_k w_k(x)$

$w_{mid}(x) = \sum_{k \in I_{mid}} a_k w_k(x)$

$y(x) = \sum_{k=0}^{m} a_k y_k(x)$

$y_{mid}(x) = \sum_{k \in I_{mid}} a_k y_k(x)$

$h(x) = \frac{(v(x)w(x) - y(x))}{t(x)}$

$r, u \leftarrow\$ \mathbb{Z}_p^*$

$f_{mid} = \frac{\beta v_{mid}(s) + \alpha w_{mid}(s) + y_{mid}(s)}{\delta}$

$a = \alpha + v(s) + r\delta$

$b = \beta + w(s) + u\delta$

$c = f_{mid} + \frac{t(s)h(s)}{\delta} + ua + rb - ur\delta$

return $\pi = (A = g^a, B = g^b, C = g^c)$

$\underline{\text{Groth.}\mathsf{Verify}(\mathsf{vk}, u, \pi)}$

$\pi = (A, B, C)$

$v_{io}(x) = \sum_{i=0}^{\ell} a_i v_i(x)$

$w_{io}(x) = \sum_{i=0}^{\ell} a_i w_i(x)$

$y_{io}(x) = \sum_{i=0}^{\ell} a_i y_i(x)$

$f_{io} = \frac{\beta v_{io}(s) + \alpha w_{io}(s) + y_{io}(s)}{\gamma}$

Check

$e(A, B) = e(g^\alpha, h^\beta) \cdot e(g^{f_{io}}, h^\gamma) \cdot e(C, h^\delta)$

$\underline{\text{Groth.}\mathsf{Sim}(\mathsf{td}, u)}$

$a, b \leftarrow\$ \mathbb{Z}_p^*$

$A = g^a, \ B = g^b$

$c = \frac{ab - \alpha\beta - \beta v_{io}(s) + \alpha w_{io}(s) + y_{io}(s)}{\delta}$

return $\pi = (A = g^a, B = g^b, C = g^c)$

Figure 1: Groth16 Construction from QAP.

Remark that for the verification algorithm, we do not use the entire structured reference string crs, but just part of it. For the sake of presentation, we will call the verifier key vk and set it using the necessary elements from the crs:

$$\mathsf{vk} := \left(P = g^\alpha, Q = h^\beta, \ \left\{ S_j = g^{\frac{\beta v_j(s) + \alpha w_j(s) + y_j(s)}{\gamma}} \right\}_{j=0}^{\ell}, H = h^\gamma, D = h^\delta \right)$$

3.1.1 Building Blocks

SRS. We need elements from two independent compatible Groth16 SRS:

- Common Bilinear group description for both SRS: $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T)$
- Common group generators for both SRS: $g \in \mathbb{G}_1, h \in \mathbb{G}_2$

- First SRS with random evaluation point $a \in \mathbb{Z}_p$ for :

$$\mathbf{v}_1 = (h, h^a, \ldots, h^{a^{n-1}}) \text{ and } \mathbf{w}_1 = (g^{a^n}, \ldots, g^{a^{2n-1}})$$

- Second SRS with random evaluation point $b \in \mathbb{Z}_p$ for :

$$\mathbf{v}_2 = (h, h^b, \ldots, h^{b^{n-1}}) \text{ and } \mathbf{w}_2 = (g^{b^n}, \ldots, g^{b^{2n-1}})$$

Commitment Schemes. Allong the Aggregation protocol, we need to commit to vectors $\mathbf{A} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n$ where $n \in \mathbb{N}$ is a power of 2.

We require the following properties from our commitment schemes:

Computationally Binding Commitment: as per Definition 2.4

Constant Size Commitment: the commitment value is independent of the length of the committed vector (two target group elements in our case)

Doubly-Homomorphic: homomorphic both in the message space and in the key space

$$\mathsf{CM}(\mathsf{ck}_1 + \mathsf{ck}_2; M_1 + M_2) = \mathsf{CM}(\mathsf{ck}_1; M_1) + \mathsf{CM}(\mathsf{ck}_1; M_2) + \mathsf{CM}(\mathsf{ck}_2; M_1) + \mathsf{CM}(\mathsf{ck}_2; M_2).$$

Collapsing Property: double-homomorphism implies a distributive property between keys and messages that allow to collapse multiple messages via a deterministic function $\mathsf{Collapse}$ defined as follows:

$$\mathsf{Collapse}\left(\mathsf{CM}\left(\begin{array}{c|c} \mathsf{ck}_1 \| \mathsf{ck}_1' & M_1 \| M_1 \\ \mathsf{ck}_2 \| \mathsf{ck}_2' & M_2 \| M_2 \\ \mathsf{ck}_3 & M_3 \end{array}\right) = \mathsf{CM}\left(\begin{array}{c|c} \mathsf{ck}_1 + \mathsf{ck}_1' & M_1 \\ \mathsf{ck}_2 + \mathsf{ck}_2' & M_2 \\ \mathsf{ck}_3 & M_3 \end{array}\right)\right)$$

Inner Product Commitments. To instantiate our commitments, we use pairing commitment schemes inspired by the ones of [LMR19]. These schemes required structured commitment keys $\mathsf{ck}_s, \mathsf{ck}_d$ of the form $\mathsf{ck}_s = (\mathbf{v}_1, \mathbf{v}_2), \mathsf{ck}_d = (\mathbf{v}_1, \mathbf{w}_1, \mathbf{v}_2, \mathbf{w}_2)$. We then commit to vectors $\mathbf{A} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n$ as follows:

1. Single group version $\mathsf{CM}_s(\mathbf{A}) \coloneqq \mathsf{CM}_s((\mathbf{v}_1, \mathbf{v}_2); \mathbf{A}) = (T_A, U_A)$ where

$$T_A = \mathbf{A} * \mathbf{v}_1 = e(A_1, h^a) \ldots . e(A_n, h^{a^n})$$
$$U_A = \mathbf{A} * \mathbf{v}_2 = e(A_1, h^b) \ldots . e(A_n, h^{b^n})$$

2. Double group version $\mathsf{CM}_d(\mathbf{A}, \mathbf{B}) \coloneqq \mathsf{CM}_d((\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2); \mathbf{A}, \mathbf{B}) = (T_{AB}, U_{AB})$ where

$$T_{AB} = (\mathbf{A} * \mathbf{v_1})(\mathbf{w_1} * \mathbf{B}) = e(A_1, h^a) \cdot \ldots e(A_n, h^{a^n}) \cdot e(g^{a^{n+1}}, B_1) \cdot \ldots e(g^{a^{2n}}, B_n)$$
$$U_{AB} = (\mathbf{A} * \mathbf{v_2})(\mathbf{w_2} * \mathbf{B}) = e(A_1, h^b) \cdot \ldots e(A_n, h^{b^n}) \cdot e(g^{b^{n+1}}, B_1) \cdot \ldots e(g^{b^{2n}}, B_n)$$

GIPA Protocols. One of the key building blocks for our aggregation protocol are *generalized inner product arguments*, called GIPA protocols.

These protocols, as designed in [BMM+19], enable proving the correctness a large class of inner products between vectors and/or field elements committed using (possibly distinct) doubly-homomorphic commitment schemes.

GIPA schemes have logarithmic communication but linear verifier time as computing the final commitment key takes a linear number of operations. These schemes use structured references string as commitment keys. Their construction is based on doubly-homomorphic commitment schemes and the KZG polynomial commitment [KZG10]. We restate the relations for the two specialized GIPA constructions from [BMM+19] bellow:

Multiexponentiation Inner Product Proof (MIPP).

The relation for MIPP is defined by :

$$\mathcal{R}_{\mathsf{MIPP}} \coloneqq \{((T_A, U_A), A, b; \mathbf{A}, \mathbf{b}) : Z = \mathbf{A}^{\mathbf{b}} \ \wedge \ (T_A, U_A) = \mathsf{CM}_s(\mathbf{A})\}.$$

Target Inner Pairing Product Proof (TIPP). A TIPP allows a prover to demonstrate that certain pairing relations hold between committed group elements.

More precisely, the relation for the TIPP we need in Groth16 aggregation is defined by:

$$\mathcal{R}_{\mathsf{TIPP}} \coloneqq \left\{((T_{AB}, U_{AB}), Z, r; \mathbf{A}, \mathbf{B}, \mathbf{r}) : Z = \mathbf{A}^{\mathbf{r}} * \mathbf{B} \wedge (T_{AB}, U_{AB}) = \mathsf{CM}_d(\mathbf{A}, \mathbf{B}) \wedge \mathbf{r} = (r^i)_{i=1}^n\right\}$$

3.2 Aggregation of Groth16

An Argument for Aggregation is a proof system that takes as input multiple proofs and computes a new smaller proof, in this case for n initial proofs we end up with a final aggregated proof of size $\mathcal{O}(\log n)$.

Overview of the protocol. The high-level idea of Groth16 aggregation is quite simple: Since Groth16 verification consists in checking a pairing equation between the proof elements $\pi = (A, B, C)$, instead of checking that n pairing equations are simultaneously satisfied it is sufficient to prove that only one inner pairing product of a random linear combination of these initial equations defined by a verifier's random challenge $r \in \mathbb{Z}_p$ holds. In a bit more detail, Groth16 verification asks to check an equation of the type $e(A_i, B_i) = Y_i \cdot e(C_i, D)$ for $Y_i \in \mathbb{G}_T, D \in \mathbb{G}_2$ where Y_i is a value computed from each statement $u_i = \mathbf{a}_i$ and $\pi_i = (A_i, B_i, C_i)_{i=1}^n$ are proof triples.

The aggregation will instead check a single randomized equation:

$$\prod_{i=1}^{n} e(A_i, B_i)^{r^i} = \prod_{i=1}^{n} Y_i' \cdot e\big(\prod_{i=1}^{n} C_i^{r^i}, D\big)$$

where Y_i' for $i = 1, \dots n$ are values computed from Y_i and randomness r^i.

This can be rewritten using an inner product notation as :

$$Z_{AB} = Y'_{prod} \cdot e(Z_C, D), \quad \text{and} \quad Z_{AB} \coloneqq \mathbf{A}^{\mathbf{r}} * \mathbf{B} \quad \text{and} \quad Z_C \coloneqq \mathbf{C} * \mathbf{r}$$

where we denoted by $Y'_{prod} \coloneqq \prod_{i=1}^{n} Y_i'$.

And what is left after checking that such an unified equation holds is to verify that the elements Z_{AB}, Z_C are consistent with the initial proof triples in the sense that they compute the required inner product. This is done by combining pairing commitments schemes and

the TIPP arguments in order to allow for showing $Z_{AB} = \mathbf{A}^{\mathbf{r}} * \mathbf{B}$ for some initial vectors $\mathbf{A} \in \mathbb{G}_1, \mathbf{B} \in \mathbb{G}_2$ committed using CM_d and MIPP argument for showing $Z_C = \mathbf{C} * \mathbf{r}$ for some vector $\mathbf{C} \in \mathbb{G}_1$ committed under CM_s.

Our scheme is non-interactive and uses a hash function Hash modeled as a random oracle. We will consider the description of this hash function publicly available for prover and verifier and part of their keys.

Relation for Aggregation. More formally, we introduce the relation for aggregation of n triplets of Groth16 proofs $\mathbf{A}, \mathbf{C} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n$:

$$\mathcal{R}_{\mathsf{AGG}} \coloneqq \left\{(\mathsf{vk}, \mathsf{pk}_{\mathsf{agg}}, \mathbf{u} = \{\mathbf{a}_i\}_{i=1}^n; \pi = \{(\mathbf{A}, \mathbf{B}, \mathbf{C})\}) : \ \text{Groth.}\mathsf{Verify}(\mathsf{vk}, u_i, \pi_i) = 1, \ \forall i \in [n]\right\}$$

where $u_i = \mathbf{a}_i = \{a_{i,j}\}_{j=0}^{\ell}, \pi_i = (A_i, B_i, C_i) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_1$ for $i = 1, \ldots n$.

3.2.1 Setup Algorithm Setup

Inputs: $(1^\lambda, \mathcal{R}_{\mathsf{AGG}})$

1. Set commitment keys for both single and double commitment schemes using crs:

$$\mathsf{ck}_s = (\mathbf{v}_1, \mathbf{v}_2), \ \mathsf{ck}_d = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2).$$

   Recall the structure of vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{G}_2$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{G}_1$:

$$\begin{aligned} \mathbf{v}_1 &= (h, h^a, \ldots, h^{a^{n-1}}), \qquad & \mathbf{w}_1 &= (g^{a^n}, \ldots, g^{a^{2n-1}}) \\ \mathbf{v}_2 &= (h, h^b, \ldots, h^{b^{n-1}}), \qquad & \mathbf{w}_2 &= (g^{b^n}, \ldots, g^{b^{2n-1}}) \end{aligned}$$

2. Construct MIPP and TIPP SRS: $\mathsf{crs}_{\mathsf{MIPP}}, \mathsf{crs}_{\mathsf{TIPP}}$
3. Choose a hash function $\mathsf{Hash} : \mathbb{G}_T^4 \to \mathbb{Z}_p$ given by its description hk.

Output: Keys: $\mathsf{pk}_{\mathsf{agg}} = (\mathsf{vk}, \mathsf{crs}_{\mathsf{MIPP}}, \mathsf{crs}_{\mathsf{TIPP}}, \mathsf{ck}_s, \mathsf{ck}_d, \mathsf{hk}), \ \mathsf{vk}_{\mathsf{agg}} = (\mathsf{vk}, \mathsf{crs}_{\mathsf{MIPP}}, \mathsf{crs}_{\mathsf{TIPP}}, \mathsf{hk})$

3.2.2 Prover Algorithm

Inputs: $\mathsf{pk}_{\mathsf{agg}}, \mathbf{u} = \{a_{i,j}\}_{i=1,\ldots n; j=0,\ldots \ell}, \ \pi = \{\pi_i\}_{i=1}^n = (\mathbf{A}, \mathbf{B}, \mathbf{C})$

1. Parse proving key $\mathsf{pk}_{\mathsf{agg}} \coloneqq (\mathsf{vk}, \mathsf{crs}_{\mathsf{MIPP}}, \mathsf{crs}_{\mathsf{TIPP}}, \mathsf{ck}_s, \mathsf{ck}_d, \mathsf{hk})$
2. Parse $\mathsf{ck}_s = (\mathbf{v}_1, \mathbf{v}_2), \ \mathsf{ck}_d = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2)$
3. Commit to $\mathbf{A}$ and $\mathbf{B}$: $\mathsf{CM}_d((\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}); \mathbf{A}, \mathbf{B}) = (T_{AB}, U_{AB})$
4. Commit to $\mathbf{C}$: $\mathsf{CM}_s((\mathbf{v_1}, \mathbf{v_2}); \mathbf{C}) = (T_C, U_C)$
5. Derive random challenge $r = \mathsf{Hash}(T_{AB}, U_{AB}, T_C, U_C) \in \mathbb{Z}_p$ and set $\mathbf{r} = \{r^i\}_{i=1}^n$
6. Compute $Z_{AB} = \langle \mathbf{A}, \mathbf{B} \rangle = \mathbf{A}^{\mathbf{r}} * \mathbf{B}$
7. Compute $Z_C = \langle \mathbf{C}, \mathbf{r} \rangle = \prod_{i=1}^n C_i^{r_i}$.

8. Compute $\mathbf{v'_1} = \mathbf{v_1^{r^{-1}}}$ and $\mathbf{v'_2} = \mathbf{v_2^{r^{-1}}}$
9. Compute TIPP proof for commitment key $((\mathbf{v'_1}, \mathbf{v'_2}), (\mathbf{w_1}, \mathbf{w_2}), 1_{\mathbb{G}_T})$:

$$\pi_{\mathsf{TIPP}} = \mathsf{TIPP.Prove}(\mathsf{crs}_{\mathsf{TIPP}}, (T_{AB}, U_{AB}), Z_{AB}, r; \mathbf{A}, \mathbf{B}, \mathbf{r})$$

10. Compute MIPP proof for commitment key $((\mathbf{w_1}, \mathbf{w_2}), 1_{\mathbb{G}_1}, 1_{\mathbb{G}_1})$:

$$\pi_{\mathsf{MIPP}} = \mathsf{MIPP.Prove}(\mathsf{crs}_{\mathsf{MIPP}}, (T_C, U_C), Z_C, r; \mathbf{C}, \mathbf{r})$$

Output: Aggregated proof $\pi_{\mathsf{agg}} = ((T_{AB}, U_{AB}), (T_C, U_C), Z_{AB}, Z_C, \pi_{\mathsf{TIPP}}, \pi_{\mathsf{MIPP}})$

3.2.3 Verification Algorithm

Inputs: $\mathsf{vk}_{\mathsf{agg}}, \mathbf{u} = \{a_{i,j}\}_{i=1,\dots n; j=0,\dots \ell}, \pi_{\mathsf{agg}}$

1. Parse verification key $\mathsf{vk}_{\mathsf{agg}} := (\mathsf{vk}, \mathsf{crs}_{\mathsf{MIPP}}, \mathsf{crs}_{\mathsf{TIPP}}, \mathsf{hk})$
2. Parse $\mathsf{vk} := \big(P = g^{\alpha}, Q = h^{\beta},\ \{S_j = g^{\frac{\beta v_j(s) + \alpha w_j(s) + y_j(s)}{\gamma}}\}_{j=0}^{\ell}, H = h^{\gamma}, D = h^{\delta}\big)$
3. Derive random challenge $r = \mathsf{Hash}(T_{AB}, U_{AB}, T_C, U_C)$
4. Check TIPP proof $\mathsf{TIPP.Verify}(\mathsf{crs}_{\mathsf{TIPP}}, (\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}), \pi_{\mathsf{TIPP}})$
5. Check MIPP proof $\mathsf{MIPP.Verify}(\mathsf{crs}_{\mathsf{MIPP}}, (\mathbf{v_1}, \mathbf{v_2}), \pi_{\mathsf{MIPP}})$
6. Compute $Z_{S_j} = \prod_{i=1}^{n} S_j^{a_{ij} r_i}$ for all $j = 0 \dots \ell$.
7. Check Groth16 aggregated equations:

$$Z_{AB} = e(P^{\sum_{i=1}^{n} r^i}, Q) e(\prod_{j=0}^{\ell} Z_{S_j}, H) e(Z_C, D)$$

Output: Decision bit $b =$ (Check TIPP) $\wedge$ (Check MIPP) $\wedge$ (Check Groth16)

4 Building Blocks Instantiations

4.1 Inner Product Pair Commitments

In this section we are looking for a commitment scheme to group elements in a bilinear group that can be compatible with the GIPA protocol described in [BMM+19]. Our goal is to find such a commitment scheme that uses a structured reference string similar to the one used in many popular SNARK implementations, e.g. Groth16.

The commitment scheme proposed by [BMM+19] works under some new assumption that implies the SRS is structured in a specific way. In order to use this commitment, we need to be careful to not give out certain elements which are present in most modern SNARK setup ceremonies.

Our commitment scheme can be used with two compatible (independent) SNARK setup ceremonies and therefore can be easily deployed without requiring a new trust assumption.

We require the two ceremonies are using the same basis in the billinear group, but two different randomnesses. Our idea is to adapt the commitment scheme of Lai, Molavolto and Ronge [LMR19] to a binding scheme with double-homomorphic properties. Instead of using GDLR assumption from [LMR19] that gives binding only for unstructured random public parameters, we introduce a new assumption that holds in the generic group model and holds when the adversary has access to structured public parameters.

SRS We need the following SRS:

1. Generators $g \in \mathbb{G}_1, h \in \mathbb{G}_2$
2. Elements related to a random $a \in \mathbb{Z}_p$: $\mathbf{v}_1 = (h, h^a, \ldots, h^{a^{n-1}})$ and $\mathbf{w}_1 = (g^{a^n}, \ldots, g^{a^{2n-1}})$
3. Elements related to a random $b \in \mathbb{Z}_p$: $\mathbf{v}_2 = (h, h^b, \ldots, h^{b^{n-1}})$ and $\mathbf{w}_2 = (g^{b^n}, \ldots, g^{b^{2n-1}})$

Construction from 2 powers of tau The construction comes naturally from 2 power of tau that used the same generators g and h, they will both have different powers $a = \tau_1$ and $b = \tau_2$: $g, h, g^{\tau_1}, \ldots, g^{\tau_1^n}, h^{\tau_1}, \ldots, h^{\tau_1^n}, g^{\tau_2}, \ldots, g^{\tau_2^n}, h^{\tau_2}, \ldots, h^{\tau_2^n}$

Our assumptions rely on the fact that cross powers (e.g. $g^{\tau_1\tau_2}$) are not known to the prover. Since the two SRS we use are the result of two independent ceremonies, it is unlikely that such terms can be learned since τ_1 and τ_2 were destroyed after the SRS generation.

For sake of efficiency, we use two commitment schemes, one that takes a vector of elements of a single group and one that takes two vectors of points in $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively.

4.1.1 Single group version CM_s

$\mathsf{KG}(1^\lambda) \to \mathsf{ck}_s = (\mathbf{v}_1, \mathbf{v}_2)$

$\mathsf{Com}(\mathsf{ck}_s = (\mathbf{v}_1, \mathbf{v}_2), \mathbf{A} = (A_1, \ldots, A_n) \in \mathbb{G}_1^n) \to (T_A, U_A)$:

1. $T_A = \mathbf{A} * \mathbf{v}_1 = e(A_1, h^a) \ldots . e(A_n, h^{a^n}) \ \in \mathbb{G}_T$
2. $U_A = \mathbf{A} * \mathbf{v}_2 = e(A_1, h^b) \ldots . e(A_n, h^{b^n}) \ \in \mathbb{G}_T$
3. For the sake of presentation, we will use the notation $\mathsf{CM}_s((\mathbf{v}_1, \mathbf{v}_2); \mathbf{A}) = (T_A, U_A)$

Lemma 4.1. *Under the hardness of q-ASSGP assumption for $q = n$, this commitment scheme is computationally binding as per Definition 2.4.*

Proof. Suppose there exists a PPT adversary $\mathcal{A}$ that breaks the binding property of the commitment scheme. Then, given the output $((T_A, U_A); \mathbf{A}, \mathbf{A}^*)$ of the adversary $\mathcal{A}$ we have that $(T_A, U_A) = (T_{A^*}, U_{A^*})$:

$$e(A_1, h^a) \ldots e(A_n, h^{a^n}) = e(A_1^*, h^a) \ldots e(A_n^*, h^{a^n}) \tag{1}$$

$$e(A_1, h^b) \ldots e(A_n, h^{b^n}) = e(A_1^; h^b) \ldots e(A_n^; h^{b^n}) \tag{2}$$

By applying the homomorphic properties of the commitment scheme to these equations we get:

$$e(A_1/A_1^*, h^a) \ldots e(A_n/A_n^*, h^{a^n}) = 1 \tag{3}$$

$$e(A_1/A_1^*, h^b) \ldots e(A_n/A_n^*, h^{b^n}) = 1 \tag{4}$$

where the vector $(A_1/A_1^*, \ldots A_n/A_n^*) \neq \mathbf{1}_{\mathbb{G}_1}$. This breaks the n-ASSGP assumption. □

4.1.2 Double group version CM_d

This one is useful for the TIPP argument used during Groth16 aggregation.

$\mathsf{KG}(1^\lambda) \to \mathsf{ck}_d = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2)$

$\mathsf{Com}(\mathsf{ck}_d, \mathbf{A} \in \mathbb{G}_1^n, \mathbf{B} \in \mathbb{G}_2^n) \to (T_{AB}, U_{AB})$:

1. $T_{AB} = (\mathbf{A} * \mathbf{v_1})(\mathbf{w_1} * \mathbf{B}) = e(A_1, h^a) \cdot \ldots e(A_n, h^{a^n}) \cdot e(g^{a^{n+1}}, B_1) \cdot \ldots e(g^{a^{2n}}, B_n) \in \mathbb{G}_T$
2. $U_{AB} = (\mathbf{A} * \mathbf{v_2})(\mathbf{w_2} * \mathbf{B}) = e(A_1, h^b) \cdot \ldots e(A_n, h^{b^n}) \cdot e(g^{b^{n+1}}, B_1) \cdot \ldots e(g^{b^{2n}}, B_n) \in \mathbb{G}_T$
3. We write $\mathsf{CM}_d((\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}); \mathbf{A}, \mathbf{B}) = (T_{AB}, U_{AB})$

Lemma 4.2. *Under the hardness of q-ASDGP assumption for $q = n$, this commitment scheme is computationally binding.*

Proof. The proof is analogous to the one of lemma 4.1. Since the commitment is homomorphic breaking the binding is equivalent to finding a non-trivial opening to 1. Thus it breaks the assumption.

□

4.2 Generalized Inner Product Arguments (GIPA)

These are the schemes designed to prove generalizations of the inner product. For completeness, we describe the framework used by [BMM+19] to build GIPA for two different types of inner product. A main observation is that both TIPP and MIPP protocols are described with respect to doubly-homomorphic inner product commitment schemes such that the inner product map is well-defined over their message space. For our instantiations, we will use the commitment schemes introduced in Section 4.1 which satisfy the desired properties for the two inner pairing product protocols.

We present the two protocols TIPP and MIPP each for a different generalization of the inner product. The two generalized inner product maps for bilinear group $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e)$ that we consider are defined by the following operations:

1. Multiexponentiation inner product map $\mathbb{G}_1^m \times \mathbb{F}^m \to \mathbb{G}_1$:

$$\mathbf{A} * \mathbf{b} \coloneqq \prod A_i^{b_i}$$

2. Inner pairing product map $\mathbb{G}_1^m \times \mathbb{G}_2^m \to \mathbb{G}_T$:

$$\mathbf{A} * \mathbf{B} \coloneqq \prod e(A_i, B_i) = \prod e(g^{a_i}, g^{b_i}) = e(g, g)^{\sum a_i b_i}$$

where e is the standard pairing map from $\mathsf{gk} : e : \mathbb{G}_1^m \times \mathbb{G}_2^m \to \mathbb{G}_T$

Commitment Schemes: For the sake of simplicity, we use a "merged" commitment that can be applied to the vectors $\mathbf{A}$ and $\mathbf{B}$ and the result of the inner product $Z = \langle \mathbf{A^r}, \mathbf{B} \rangle$ at once with a composed commitment key defined as $\mathsf{ck} = (\mathsf{ck}_1, \mathsf{ck}_2, \mathsf{ck}_3)$.

We recall the doubly-homomorphic property of the commitment scheme in multiplicative notations:

1. $\mathsf{CM}(\mathsf{ck}, M) \cdot \mathsf{CM}(\mathsf{ck}, M') = \mathsf{CM}(\mathsf{ck}, M \cdot M')$
2. $\mathsf{CM}(\mathsf{ck}, M) \cdot \mathsf{CM}(\mathsf{ck}', M) = \mathsf{CM}(\mathsf{ck} \cdot \mathsf{ck}', M)$
3. (Follows from 1&2) $\mathsf{CM}(\mathsf{ck}^x, M) = \mathsf{CM}(\mathsf{ck}, M^x)$, where $x \in \mathbb{Z}_p$.

Polynomial Commitment: We will need a polynomial commitment scheme (Definition 2.2.3) that allows for openings of evaluations on a point and proving correctness of these openings. Specifically we need the polynomial commitments to prove *succinctly* the correctness of the final commitment keys v_1, v_2 and w_1, w_2 at the end of the protocol. We can use a polynomial commitment scheme here because these final commitment keys have a well defined structure as shown in [BMM+19]. The candidate for the PC is KZG scheme in [KZG10] which allows us to have a constant time verifier for this check.

$\mathsf{KZG.PC} = (\mathsf{KZG.KG}, \mathsf{KZG.CM}, \mathsf{KZG.Open}, \mathsf{KZG.Check})$ defined over billinear groups $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T)$ with $\mathbb{G}_1 = \langle g \rangle, \mathbb{G}_2 = \langle h \rangle$ as follows:

$\mathsf{KZG.KG}(1^\lambda, n) \to (\mathsf{ck}_{\mathsf{KZG}}, \mathsf{vk}_{\mathsf{KZG}})$**:** Set keys $\mathsf{ck}_{\mathsf{KZG}} = \{g^{\alpha^i}\}_{i=0}^{n-1}, \mathsf{vk}_{\mathsf{KZG}} = h^\alpha$.

$\mathsf{KZG.CM}(\mathsf{ck}_{\mathsf{KZG}}; f(X)) \to C_f$**:** For $f(X) = \sum_{i=0}^{n-1} f_i X^i$, computes $C_f = \prod_{i=0}^{n-1} g^{f_i \alpha^i} = g^{f(\alpha)}$.

$\mathsf{KZG.Open}(\mathsf{ck}_{\mathsf{KZG}}; C_f, x, y; f(X)) \to \pi$**:** For an evaluation point x, a value y, compute the quotient polynomial $q(X) = \dfrac{f(X) - y}{X - x}$ and output prove $\pi = C_q = \mathsf{KZG.CM}(\mathsf{ck}_{\mathsf{KZG}}; q(X))$.

$\mathsf{KZG.Check}(\mathsf{vk}_{\mathsf{KZG}} = h^\alpha, C_f, x, y, \pi) \to 1/0$**:** Check if $e(C_f \cdot g^{-y}, h) = e(C_q \cdot g^{-x}, h^\alpha)$.

The $\mathsf{KZG.PC}$ scheme works in a similar fashion for a pair of keys of the form $\mathsf{ck}_{\mathsf{KZG}} = \{h^{\alpha^i}\}_{i=0}^{n-1}, \mathsf{vk}_{\mathsf{KZG}} = g^\alpha$, by just swapping the values in the final pairing equation check to match the correct basis.

4.2.1 MIPP with Pair Group Commitment

This is a multiexponentiation inner product argument for the relation

$$\mathcal{R}_{\mathsf{MIPP}} := \{((T_A, U_A), A, b; \mathbf{A}, \mathbf{b}) : Z = \mathbf{A}^{\mathbf{b}} \ \wedge \ (T_A, U_A) = \mathsf{CM}_s(\mathbf{A})\}.$$

The following protocol is a variation of the original MIPP argument presented in [BMM+19]. The main difference is the pairing commitment scheme used, in our version we employ CM_s which relies on a Groth16-friendly setup ceremony as discussed in the introduction.

For the proving strategy, the idea is the same as in the MIPP from [BMM+19]: In a nutshell, the prover runs a loop and in each iteration it is first splitting the initial vector $\mathbf{A}$ in half and then using collapsing property of the commitment (see definition for $\mathsf{Collapse}$ function in building-blocks Section 3.1.1) to recommit to both halves together.

The MIPP protocol uses the identity for the $\mathsf{Collapse}$ function defined $\mathsf{Collapse}_{id}(C) = C$. Since all components of the commitment are compact, the identity collapsing function is sufficient.

After reducing the size of the commitment keys to 1, the prover has to show well-formedness of the such-obtained final structured commitment keys. The final keys v_1, v_2 are interpreted as a KZG polynomial commitment that the prover must open at a random point z.

Construction. Our MIPP argument consists in 3 algorithms $\mathsf{MIPP} = (\mathsf{MIPP.Setup}, \mathsf{MIPP.Prove}, \mathsf{MIPP.Verify})$ that work as follows:

$\mathsf{MIPP.Setup}(1^\lambda, \mathcal{R}_{\mathsf{MIPP}}) \to \mathsf{crs}_{\mathsf{MIPP}}$**:**

1. Set commitment keys for KZG scheme from ck_s (specified by the relation):

$$\mathsf{ck}_{\mathsf{1KZG}} \coloneqq \{h^{a^i}\}_{i=1}^n, \mathsf{vk}_{\mathsf{1KZG}} \coloneqq g^a$$
$$\mathsf{ck}_{\mathsf{2KZG}} \coloneqq \{h^{b^i}\}_{i=1}^n, \mathsf{vk}_{\mathsf{2KZG}} \coloneqq g^b$$

2. Fix a hash function $\mathsf{Hash}_1 : \mathbb{Z}_p \times \mathbb{G}_T^6 \to \mathbb{Z}_p$ and its description hk_1.
3. Fix a hash function $\mathsf{Hash}_2 : \mathbb{Z}_p^n \times \mathbb{G}_2^2 \times \mathbb{G}_1^2 \to \mathbb{Z}_p$ and its description hk_2.
4. Set $\mathsf{crs}_{\mathsf{MIPP}} \coloneqq (\mathsf{hk}_1, \mathsf{hk}_2, \mathsf{ck}_{\mathsf{1KZG}}, \mathsf{vk}_{\mathsf{1KZG}}, \mathsf{ck}_{\mathsf{2KZG}}, \mathsf{vk}_{\mathsf{2KZG}})$

$\mathsf{MIPP.Prove}(\mathsf{crs}_{\mathsf{MIPP}}, (T_A, U_A), Z, r; \mathbf{A}, \mathbf{r}) \to \pi_{\mathsf{MIPP}}$**:**

- Loop "split & collapse"
  1. $n' = n/2$
  2. If $n' == 1$: **break**
  3. Compute left and right inner products

$$Z_L = \mathbf{A}_{[n':]}^{\mathbf{r}_{[:n']}} = \prod_{i=0}^{n'} A_{i+n'}^{r_i}$$
$$Z_R = \mathbf{A}_{[:n']}^{\mathbf{r}_{[n':]}} = \prod_{i=0}^{n'} A_i^{r_{i+n'}}$$

  4. Compute right cross commitments:

$$T_L, U_L = CM_s((\mathbf{v}_1, \mathbf{v}_2), \mathbf{A}_{[n':]} || \mathbf{0}) = ((\mathbf{A}_{[n':]} * \mathbf{v}_{1[:n']}), (\mathbf{A}_{[n':]} * \mathbf{v}_{2[:n']}))$$

  5. Compute left cross commitments:

$$T_R, U_R = CM_s((\mathbf{v_1}, \mathbf{v_2}), \mathbf{0} || \mathbf{A}_{[:n']}) = ((\mathbf{A}_{[:n']} * \mathbf{v}_{1[n':]}), (\mathbf{A}_{[:n']} * \mathbf{v}_{2[n':]}))$$

  6. Compute challenge $x_i = \mathsf{Hash}_1(x_{i-1}, Z_L, Z_R, T_R, T_L, U_R, U_L)$ (with $x_0 = 0$)
  7. Compute Hadamard products

$$\mathbf{A} = \mathbf{A}_{[:n']} \circ \mathbf{A}_{[n':]}^x = (A_0 A_{n'}^x, \ldots, A_{n'-1} A_{n-1}^x)$$
$$\mathbf{r} = \mathbf{r}_{[:n']} \circ \mathbf{r}_{[n':]}^{x^{-1}} = (r_0 r_{n'}^{x^{-1}}, \ldots, r_{n'-1} r_{n-1}^{x^{-1}})$$

  8. Set new collapsed key $(\mathbf{v}_1, \mathbf{v}_2) \coloneqq (\mathbf{v_1}_{[:n']} \circ \mathbf{v_1}_{[n':]}^{x^{-1}}, \mathbf{v_2}_{[:n']} \circ \mathbf{v'_2}_{[n':]}^{x^{-1}})$

- Prove correctness of final commitment key $(v_1, v_2) \in \mathbb{G}_2^2$:
  1. Define $f_v(X) = \prod_{i=0}^{l-1}(1 + x_{l-j}^{-1} X^{2^j})$
  2. Draw challenge $z = \mathsf{Hash}_2(\mathbf{x}, v_1, v_2)$ (from all challenges $\mathbf{x}$ and (v_1, v_2))
  3. Prove that $v_1 = (h^a)^{f_v(a)}$ and $v_2 = (h^b)^{f_v(b)}$ are KZG commitments of $f_v(X)$ by evaluation in z

$$\pi_{v_1} \leftarrow \mathsf{KZG.Open}(\mathsf{ck}_{1\mathsf{KZG}}; v_1, z, f_v(z); f_v(X))$$
$$\pi_{v_2} \leftarrow \mathsf{KZG.Open}(\mathsf{ck}_{2\mathsf{KZG}}; v_2, z, f_v(z); f_v(X))$$

  4. Set $\pi_v = (\pi_{v_1}, \pi_{v_2})$
- For A and r' (elements from the last step of the loop with respect $\mathbf{A}$ and to $\mathbf{r}$) set

$$\pi_{\mathsf{MIPP}} = (A, r', \mathbf{Z_L}, \mathbf{Z_R}, \mathbf{T_L}, \mathbf{T_R}, \mathbf{U_L}, \mathbf{U_R}, (v_1, v_2), \pi_v)$$

$\mathsf{MIPP.Verify}(\mathsf{crs}_{\mathsf{MIPP}}, (T_A, U_A), Z, r; \pi_{\mathsf{MIPP}}) \rightarrow b$**:**

- Loop iterator $i : 1 \rightarrow n$:
  1. Reconstruct challenges $\{x_i = H(x_{i-1}, \mathbf{Z_L}[i], \mathbf{Z_R}[i], \mathbf{T_L}[i], \mathbf{T_R}[i], \mathbf{U_L}[i], \mathbf{U_R}[i]\}_{i=1}^{log(n)}\}_{i=0}^{n}$ with $x_0 = 0$
  2. Construct final commitment values:
     - $Z = \mathbf{Z_L}[i]^{x_i} \cdot Z \cdot \mathbf{Z_R}[i]^{x_i^{-1}}$ (representing $\mathsf{CM}_{id}(\mathbf{1}_{\mathbb{G}_2}; \mathbf{A^r})$)
     - $T_A = \mathbf{T_L}[i]^{x_i} \cdot T \cdot \mathbf{T_R}[i]^{x^{-1}}, \qquad U = \mathbf{U_L}[i]^{x_i} \cdot U_A \cdot \mathbf{U_R}[i]^{x^{-1}}$ (representing $\mathsf{CM}_s((\mathbf{v_1}, \mathbf{v_2}); \mathbf{A})$)
- Verify commitments into decisional bit b_0:
  1. $Z == A^{r'}$
  2. $T == e(A, v_1), \quad U == e(A, v_2)$
  3. Check if $(e(v_1, a) == T_A$ and $e(v_2, a)) == U_A$
- Verify final commitment keys $\mathbf{v_1}, \mathbf{v_2}$ via KZG
  1. Reconstruct KZG challenge point: $z = H(x_{log(n)}, v_1, v_2)$
  2. Reconstruct commitment polynomial $f_v(X) = \prod_{i=0}^{l-1}(1 + x_{l-j}^{-1} X^{2^j})$
  3. Run verification for openings of evaluations in z

$$b_1 \leftarrow \mathsf{KZG.Check}(\mathsf{vk}_{1\mathsf{KZG}}; v_1, z, f_v(z); \pi_{v_1})$$
$$b_2 \leftarrow \mathsf{KZG.Check}(\mathsf{vk}_{2\mathsf{KZG}}; v_2, z, f_v(z); \pi_{v_2})$$

  4. Set $b = b_0 \ \wedge b_1 \ \wedge b_2$

The security result for the MIPP protocol is following the same arguments as the one in [BMM+19]:

Theorem 4.3. *If* CM_s *is a binding inner product commitment,* $\mathsf{KZG.PC}$ *is a polynomial commitment with Computational Knowledge Binding as per Definition 2.6, then the protocol MIPP has computational knowledge soundness (Definition 2.1).*

Remark that both CM_s and $\mathsf{KZG.PC}$ schemes are secure in the Generic Group Model (or under specific assumptions such as q-ASSGP and q-SDH).

4.2.2 TIPP with Pair Group Commitment

This is an inner pairing argument for the relation:

$$\mathcal{R}_{\mathsf{TIPP}} := \left\{((T_{AB}, U_{AB}), Z, r; \mathbf{A}, \mathbf{B}, \mathbf{r}) : Z = \mathbf{A}^{\mathbf{r}} * \mathbf{B} \ \wedge \ (T_{AB}, U_{AB}) = \mathsf{CM}_d(\mathbf{A}, \mathbf{B}) \ \wedge \ \mathbf{r} = (r^i)_{i=0}^{n-1}\right\}.$$

where $(T_{AB}, U_{AB}) \in \mathbb{G}_T \times \mathbb{G}_T,\ Z = \mathbf{A}^{\mathbf{r}} * \mathbf{B} \in \mathbb{G}_T,\ \mathbf{A} \in \mathbb{G}_1^n,\ \mathbf{B} \in \mathbb{G}_2^n,\ r \in \mathbb{Z}_p$.

It works similarly to MIPP argument, with the difference that vectors of group elements $\mathbf{A}, \mathbf{B}$ are committed together using the double version of the pairing commitment scheme CM_d.

Construction. Our TIPP argument consists in 3 algorithms $\mathsf{TIPP} = (\mathsf{TIPP.Setup}, \mathsf{TIPP.Prove}, \mathsf{TIPP.Verify})$ that work as follows:

$\mathsf{TIPP.Setup}(1^\lambda, \mathcal{R}_{\mathsf{TIPP}}) \to \mathsf{crs}_{\mathsf{TIPP}}$**:**

1. Set commitment keys for KZG scheme from ck_d (specified by the relation):

$$\mathsf{ck}_{1v} := \{h^{a^i}\}_{i=1}^n,\ \mathsf{vk}_{1v} := g^a \qquad \mathsf{ck}_{1w} := \{g^{a^i}\}_{i=1}^n,\ \mathsf{vk}_{1w} := h^a$$
$$\mathsf{ck}_{2v} := \{h^{b^i}\}_{i=1}^n,\ \mathsf{vk}_{2v} := g^b \qquad \mathsf{ck}_{2w} := \{g^{b^i}\}_{i=1}^n,\ \mathsf{vk}_{2w} := h^b$$

2. Define $\mathsf{ck}_{\mathsf{KZG}} := (\mathsf{ck}_{j\sigma\mathsf{KZG}})\ \mathsf{vk}_{\mathsf{KZG}} := (\mathsf{vk}_{j\sigma\mathsf{KZG}})$ for $j = 1, 2;\ \sigma = v, w$
3. Fix a hash function $\mathsf{Hash}_1 : \mathbb{Z}_p \times \mathbb{G}_T^6 \to \mathbb{Z}_p$ and its description hk_1.
4. Fix a hash function $\mathsf{Hash}_2 : \mathbb{Z}_p^n \times \mathbb{G}_2^2 \times \mathbb{G}_1^2 \to \mathbb{Z}_p$ and its description hk_2.
5. Set $\mathsf{crs}_{\mathsf{TIPP}} := (\mathsf{hk}_1, \mathsf{hk}_2, \mathsf{ck}_{\mathsf{KZG}}, \mathsf{vk}_{\mathsf{KZG}})$

$\mathsf{TIPP.Prove}(\mathsf{crs}_{\mathsf{TIPP}}, (T_{AB}, U_{AB}), Z, r; \mathbf{A}, \mathbf{B}, \mathbf{r}) \to \pi_{\mathsf{TIPP}}$**:**
For ease of exposition, set $\mathbf{A} := \mathbf{A}^{\mathbf{r}}$ and $\mathbf{v_1'} := \mathbf{v_1}^{\mathbf{r}^{-1}}$ and $\mathbf{v_2'} := \mathbf{v_2}^{\mathbf{r}^{-1}}$.

- Loop "split & collapse"
  1. $n' = n/2$
  2. If $n' == 1$: **break**
  3. Compute left and right inner products

$$Z_L = \mathbf{A}_{[n':]} * \mathbf{B}_{[:n']} \ \text{ and } \ Z_R = \mathbf{A}_{[:n']} * \mathbf{B}_{[n':]}$$

  4. Compute right cross commitments:

$$T_L, U_L = \mathsf{CM}_d((\mathbf{v}_1, \mathbf{w}_1; \mathbf{v}_2, \mathbf{w}_2); \mathbf{A}_{[n':]} || \mathbf{0}, \mathbf{0} || \mathbf{B}_{[:n']}))$$
$$= ((\mathbf{A}_{[n':]} * \mathbf{v_1}_{[:n']})(\mathbf{w_1}_{[n:']} * \mathbf{B}_{[:n']}), (\mathbf{A}_{[n':]} * \mathbf{v_2}_{[:n']})(\mathbf{w_2}_{[n':]} * \mathbf{B}_{[:n']}))$$

  5. Compute left cross commitments:

$$T_R, U_R = \mathsf{CM}_d((\mathbf{v}_1, \mathbf{w}_1; \mathbf{v}_2, \mathbf{w}_2); \mathbf{0} || \mathbf{A}_{[:n']}, \mathbf{B}_{[n':]} || \mathbf{0}) =$$
$$((\mathbf{A}_{[:n']} * \mathbf{v_1}_{[n':]})(\mathbf{w_1}_{[:n']} * \mathbf{B}_{[n':]}), (\mathbf{A}_{[:n']} * \mathbf{v_2}_{[n':]})(\mathbf{w_2}_{[:n']} * \mathbf{B}_{[n':]})$$

6. Compute challenge $x_i = \mathsf{Hash}_1(x_{i-1}, Z_L, Z_R, T_R, T_L, U_R, U_L)$ (with $x_0 = 0$)
7. Compute Hadamard products on vectors

$$\mathbf{A} = \mathbf{A}_{[:n']} \circ \mathbf{A}^x_{[n':]} = (A_0 A^x_{n'}, \ldots, A_{n'-1} A^x_{n-1}) \text{ and } \mathbf{B} = \mathbf{B}_{[:n']} \circ \mathbf{B}^{x^{-1}}_{[n':]}$$

8. Compute Hadamard products on keys

$$(\mathbf{v_1}, \mathbf{v_2}) = (\mathbf{v_1}_{[:n']} \circ \mathbf{v_1}^{x^{-1}}_{[n':]}, \mathbf{v_2}_{[:n']} \circ \mathbf{v_2}^{x^{-1}}_{[n':]})$$
$$(\mathbf{w_1}, \mathbf{w_2}) = (\mathbf{w_1}_{[:n']} \circ \mathbf{w_1}^{x}_{[n':]}, \mathbf{w_2}_{[:n']} \circ \mathbf{w_2}^{x}_{[n':]})$$

9. Set $n = n'$

- Prove correctness of final commitment keys $(v_1, v_2) \in \mathbb{G}_2^2; (w_1, w_2) \in \mathbb{G}_1^2$ using KZG:
  1. Define $f_v(X) = \prod_{i=0}^{l-1}(1 + x_{l-j}^{-1} r^{-1} X)^{2^j}$ and $f_w(X) = \prod_{i=0}^{l-1}(1 + x_{l-j} X^{2^j})$
  2. Draw challenge $z = \mathsf{Hash}_2(\mathbf{x}, v_1, v_2, w_1, w_2)$
  3. Prove that $v_1 = (g^a)^{f_v(a)}$, $v_2 = (h^a)^{f_v(a)}$ and $w_1 = (g^{a^{n+1}})^{f_w(a)}$, $w_2 = (h^{a^{n+1}})^{f_{w_2}(b)}$ are KZG commitments of $f_v(X)$ by evaluating in z

$$\pi_{v_j} \leftarrow \mathsf{KZG.Open}(\mathsf{ck}_{jv}; v_j, z, f_v(z); f_v(X)) \text{ for j=1,2}$$
$$\pi_{w_j} \leftarrow \mathsf{KZG.Open}(\mathsf{ck}_{jw}; w_j, z, f_w(z); f_w(X)) \text{ for j=1,2}$$

- Set

$$\pi_{\mathsf{TIPP}} = (A, B, \mathbf{Z_L}, \mathbf{Z_R}, \mathbf{T_L}, \mathbf{T_R}, \mathbf{U_L}, \mathbf{U_R}, (v_1, v_2), (w_1, w_2), (\pi_{v_j}, \pi_{w_j})_{j=1,2})$$

where A and B are the final elements from the loop after collapsing $\mathbf{A}$ and $\mathbf{B}$.

$\mathsf{TIPP.Verify}(\mathsf{crs}_{\mathsf{TIPP}}, (T_{AB}, U_{AB}), Z, r; \pi_{\mathsf{TIPP}}) \rightarrow b$**:**

- Loop iterator $i : 1 \rightarrow n$:
  1. Reconstruct challenges $\{x_i = H(x_{i-1}, \mathbf{Z_L}[i], \mathbf{Z_R}[i], \mathbf{T_L}[i], \mathbf{T_R}[i], \mathbf{U_L}[i], \mathbf{U_R}[i])\}_{i=1}^{log(n)}$ with $x_0 = 0$
  2. Construct final commitment values recursively, $i = 1 \rightarrow n$:
     - $Z = \mathbf{Z_L}[i]^{x_i}.Z.\mathbf{Z_R}[i]^{x_i^{-1}}$
     - $T = \mathbf{T_L}[i]^{x_i}.T.\mathbf{T_R}[i]^{x^{-1}}$
     - $U = \mathbf{U_L}[i]^{x_i}.U.\mathbf{U_R}[i]^{x^{-1}}$
- Verify commitments into decisional bit b_0:
  1. $Z == e(A, B)$
  2. $T == e(A, v_1)e(w_1, B)$
  3. Check if $e(a, v_1)(e(w_1, b) == T$ and $e(a, v_2)e(w_2, b) == U$
- Verify final commitment keys $\mathbf{v_j}, \mathbf{w_j}$, for $j = 1, 2$ via KZG
  1. Reconstruct KZG challenge point: $z = H(x_{log(n)}, v_1, v_2, w_1, w_2)$

2. Reconstruct commitment polynomials:

$$f_v = \prod_{i=0}^{l-1}(1 + x_{l-j}^{-1}(r^{-1}X)^{2^j}) \tag{5}$$

$$f_w = \prod_{i=0}^{l-1}(1 + x_{l-j}X^{2^j}) \tag{6}$$

3. Run verification for openings of evaluations in z for $j = 1, 2$:

$$b_{1j} \leftarrow \mathsf{KZG.Check}(\mathsf{vk}_{jv}; v_j, z, f_v(z); \pi_{v_j})$$
$$b_{2j} \leftarrow \mathsf{KZG.Check}(\mathsf{vk}_{jw}; w_j, z, f_w(z); \pi_{w_j})$$

- Set $b = b_0 \ \wedge b_{11} \ \wedge b_{12} \ \wedge b_{21} \ \wedge b_{22}$

The security result for the TIPP protocol is following the same proving strategy as the one in [BMM+19]:

Theorem 4.4. *If CM_d is a binding inner product commitment, $\mathsf{KZG.PC}$ is a polynomial commitment with Computational Knowledge Binding as per Definition 2.6, then the protocol TIPP has computational knowledge soundness (Definition 2.1).*

Remark that both CM_d and $\mathsf{KZG.PC}$ schemes are secure in the Generic Group Model (or under specific assumptions such as q-ASDGP and q-SDH).

5 Implementation

5.1 Setup

We have implemented the scheme in Rust, using the paired [Fil18b] library on the BLS12-381 curve. The code can be found on the feat-ipp2 branch [Fil21] of the bellperson repository [Fil18a]. We have taken the original code of the arkwork library [ark19] and modified it both for fitting the scheme presented in this paper and for performance. All proofs are Groth16 proofs with 350 public inputs, which is similar to the proofs posted by Filecoin miners. All benchmarks are done on a 32 cores / 64 threads machine with AMD Raizen Threadripper CPUs.

Parallelism: It is important to note that the protocol allows for some parallel operations and our implementation makes use of that. Therefore, all benchmarks presented here can change depending on the degree of parallelism of the machine.

5.2 Trusted Setup

We created a condensed version of the SRS required for our protocol from the powers of tau transcript of both Zcash [zca18] and Filecoin [Lab18]. The code to assemble the SRS from two powers of tau can be found at [nik21]. The srs created allows to aggregate up to 2^19 proofs.

5.3 Optimization

We have implemented a further optimization to our protocol that enabled us to achieve a 20-30% improvement in verification time as well as a slighter reduction in proof size. The MIPP and TIPP protocols are very similar due to their common root which is GIPA. In particular they both follow the same structure: they use the Fiat Shamir heuristic and prove the correctness of the commitment keys with KZG proofs afterwards. The gist of the optimization is to apply the Fiat Shamir heuristic *at the same time* for both MIPP and TIPP, leading to twice less calls to the random oracle as well as one less KZG proof to verify. To do this, we had to modify both the relation of TIPP from $Z = \mathbf{A^r} * \mathbf{B}$ to $Z = \mathbf{A} * \mathbf{B^r}$ and the scaling of the commitment keys: $\mathbf{w}' = \mathbf{w}^{\mathbf{r}^{-1}}$ ($\mathbf{v}$ is not rescaled by $\mathbf{r}$ anymore). The security of the scheme is still preserved since it is simply a change of variable. This simple change however allows us to re-use the same KZG proofs for $\mathbf{v}$ in TIPP and MIPP. That optimizations enables to save 4 pairings and a logarithmic number of group multiplications.

5.4 Verification time

The major point of interest in our application to Filecoin is the verification time of Groth16 proofs. Figure 2 shows the comparison between the verification of an aggregated proof and using batching techniques as described in the zcash protocol [DH21]. Verifying Groth16 proofs in batches is what is commonly used in zcash as well as Filecoin to get a sublinear verification time. The graph shows that batching is more efficient when verifying less 100 Groth16 proofs but aggregation becomes exponentially faster after that point. Our protocol can verify a 1024 proofs in 23ms and it scales logarithmically (31ms for 8192 proofs). Note the verification algorithm is *linear* in terms of the public inputs. In our case, 350 public inputs is small enough to barely count for the total verification time.

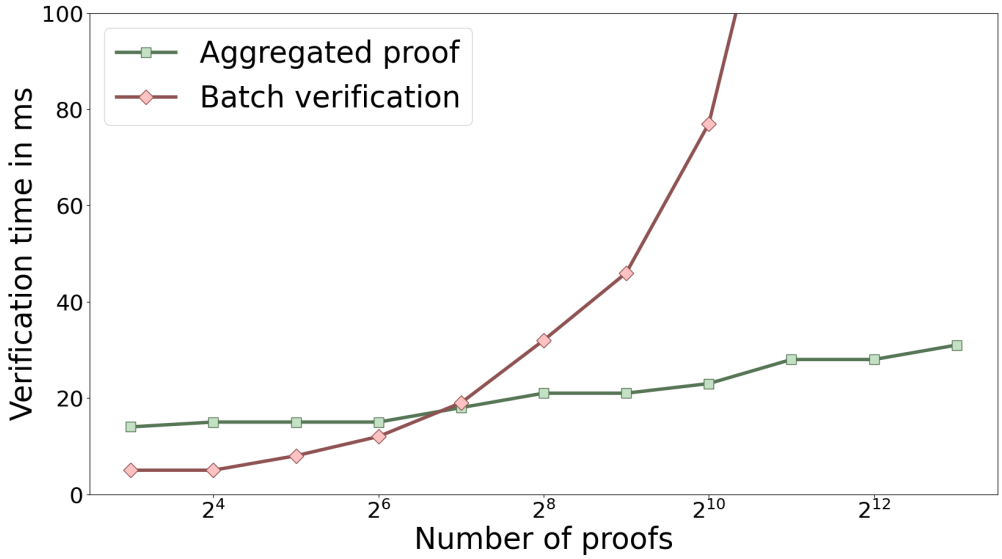

Figure 2: Verifcation time: Aggregation vs Batching.

5.5 Proof size

The proof size figure 3 compares the size of n proofs versus the size of one aggregated proof. The figure shows the break even point around 250 proofs where aggregation takes less space than batching. At 256 proofs the size is of 50kB and is only of 80kB for 8192 proofs.

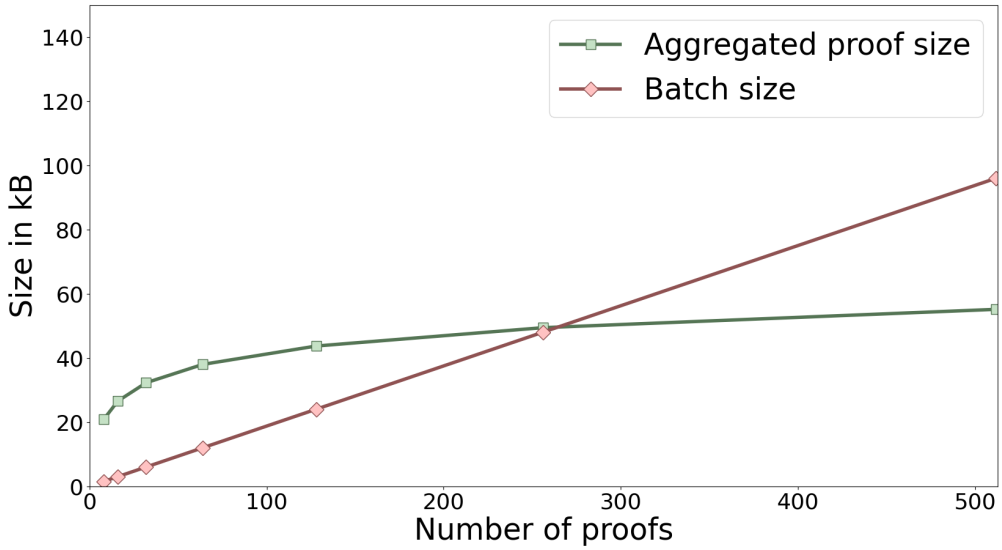

Figure 3: Proof size: Aggregation vs Batching.

5.6 Aggregation time

Figure 4 shows the time taken by the aggregator to create an aggregated proof. We can see for example that it can aggregate 1024 proofs in 2s. The prover is required to compute a logarithmic number of multi-exponentiations and expensive pairing products. Our implementation perform these in parallel and in batches (batching miller loop operations).

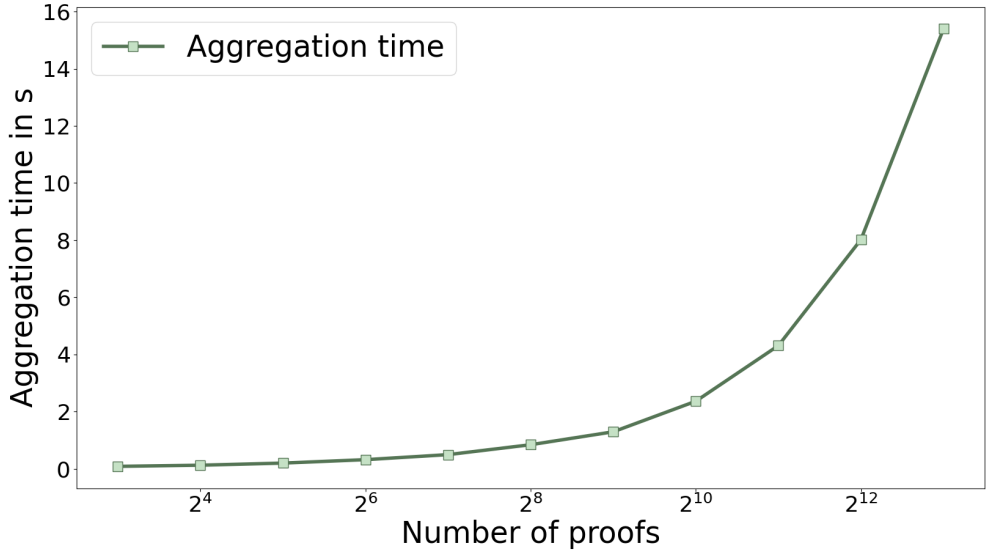

Figure 4: Aggregation Time

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A Assumptions in Generic Group Model

A.1 ASSGP Assumption in GGM

Assumption A.1 (ASSGP)**.** *The q-ASSGP assumption holds for the bilinear group generator $\mathcal{G}$ if for all* PPT *adversaries $\mathcal{A}$ we have, on the probability space* $\mathsf{gk} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T) \leftarrow \mathcal{G}(1^\lambda)$, $g \leftarrow\!\$\, \mathbb{G}_1, h \leftarrow\!\$\, \mathbb{G}_2$ *and* $a, b \leftarrow\!\$\, \mathbb{Z}_p$ *the following holds:*

$$\Pr\left[\begin{array}{c|c} (A_0,\ldots,A_{q-1}) \neq \mathbf{1}_{\mathbb{G}_1} & g \leftarrow\!\$\, \mathbb{G}_1, h \leftarrow\!\$\, \mathbb{G}_2, a, b \leftarrow\!\$\, \mathbb{Z}_p \\ \wedge \prod_{i=0}^{q-1} e(A_i, h^{a^i}) = 1_{\mathbb{G}_T} & \sigma \leftarrow ([g^{b^i}]_{i=0}^{2q-1}, [g^{a^i}]_{i=0}^{2q-1}, [h^{b^i}]_{i=0}^{2q-1}, [h^{a^i}]_{i=0}^{2q-1}) \\ \wedge \prod_{i=0}^{q-1} e(A_i, h^{b^i}) = 1_{\mathbb{G}_T} & (A_1,\ldots,A_q) \leftarrow \mathcal{A}(\mathsf{gk}, \sigma) \end{array}\right] = \mathsf{negl}(\lambda)$$

Lemma A.2. *The q-ASSGP assumption holds in the generic group model.*

Proof. Suppose $\mathcal{A}$ is an adversary that on input (gk, σ), outputs $(A_0, \ldots, A_{q-1}) \in \mathbb{G}_1^n$ such that $\prod_{i=0}^{q-1} e(A_i, h^{a^i}) = 1_{\mathbb{G}_T}$ and $\prod_{i=0}^{q-1} e(A_i, h^{b^i}) = 1_{\mathbb{G}_T}$. Then its GGM extractor outputs $\alpha_i(X, Y) = \sum_{j=0}^{2q-1}(x_j X^j + y_j Y^j + c_j)$ then we have:

$$\alpha_0(X,Y) + X\alpha_1(X,Y) + X^2\alpha_2(X,Y) + \cdots + X^{q-1}\alpha_{q-1}(X,Y) = 0 \tag{7}$$
$$\alpha_0(X,Y) + Y\alpha_1(X,Y) + Y^2\alpha_2(X,Y) + \cdots + Y^{q-1}\alpha_{q-1}(X,Y) = 0 \tag{8}$$

Then we have:

$$\alpha_0(X,Y) = -X\alpha_1(X,Y) - X^2\alpha_2(X,Y) - \cdots - X^{q-1}\alpha_{q-1}(X,Y) \tag{9}$$
$$\alpha_0(X,Y) = -Y\alpha_1(X,Y) - Y^2\alpha_2(X,Y) - \cdots - Y^{q-1}\alpha_{q-1}(X,Y) \tag{10}$$

If we substract (10) and (9) we got

$$0 = (X-Y)\alpha_1(X,Y) + \cdots + (X^{q-1} - Y^{q-1})\alpha_{q-1}(X,Y) \tag{11}$$
$$-(X-Y)\alpha_1(X,Y) = (X^2 - Y^2)\alpha_2(X,Y) + \cdots + (X^{q-1} - Y^{q-1})\alpha_{q-1}(X,Y) \tag{12}$$

Now we can divide by $(X - Y)$ and obtain:

$$\begin{aligned} -\alpha_1(X,Y) = &(X+Y)\alpha_2(X,Y) + (X^2 + XY + Y^2)\alpha_3(X,Y) + \cdots + \\ &+ (X^{q-2} + YX^{q-3} + \cdots + Y^{q-3}X + Y^{q-2})\alpha_{q-1}(X,Y) \end{aligned} \tag{13}$$

Substitute the expression of $-\alpha_1(X, Y)$ in equation (9) and remark that all $X^i\alpha_i(X, Y)$ terms are vanishing:

$$\begin{aligned} \alpha_0(X,Y) &= X[(X+Y)\alpha_2(X,Y) + (X^2 + XY + Y^2)\alpha_3(X,Y) + \cdots + (X^{q-2} + X^{q-3}Y + \cdots + \\ &\quad + XY^{q-3} + Y^{q-2})\alpha_{q-1}(X,Y)] - X^2\alpha_2(X,Y) - \cdots - X^{q-1}\alpha_{q-1}(X,Y) \\ \alpha_0(X,Y) &= XY\alpha_2(X,Y) + (X^2Y + XY^2)\alpha_3(X,Y) + \cdots + (X^{q-2}Y + \cdots + XY^{q-2})\alpha_{q-1}(X,Y) \\ \alpha_0(X,Y) &= XY[\alpha_2(X,Y) + (X+Y)\alpha_3(X,Y) + \cdots + (X^{q-3} + X^{q-4}Y + \cdots + Y^{q-3})\alpha_{q-1}(X,Y)] \end{aligned} \tag{14}$$

This implies that either $\alpha_0(X,Y)$ is a multiple of XY or $\alpha_0(X,Y)=0$.

By the GGM assumption, we have that $\alpha_0(X,Y)=0$.

We continue by replacing $\alpha_0(X,Y)=0$ in equation (14):

$$0=\alpha_2(X,Y)+(X+Y)\alpha_3(X,Y)+\cdots+(X^{q-3}+X^{q-4}Y+\cdots+Y^{q-3})\alpha_{q-1}(X,Y) \tag{15}$$

$$-\alpha_2(X,Y)=(X+Y)\alpha_3(X,Y)+\cdots+(X^{q-3}+X^{q-4}Y+\cdots+Y^{q-3})\alpha_{q-1}(X,Y) \tag{16}$$

Substitute the expression of $-\alpha_2(X,Y)$ in equation (10) and remark that all $Y^i\alpha_i(X,Y)$ terms are vanishing:

$$\begin{aligned}0=&-Y\alpha_1(X,Y)-Y^2[(X+Y)\alpha_3(X,Y)+\cdots+(X^{q-3}+X^{q-4}Y+\cdots+Y^{q-3})\alpha_{q-1}(X,Y)]-\\&-Y^3\alpha_3(X,Y)-\cdots-Y^{q-1}\alpha_{q-1}(X,Y)\\ Y\alpha_1(X,Y)=&Y^2X\alpha_3(X,Y)+\cdots+(X^{q-3}Y^2+X^{q-4}Y^3+\cdots+XY^{q-2})\alpha_{q-1}(X,Y)]\\ Y\alpha_1(X,Y)=&Y^2X[\alpha_3(X,Y)+\cdots+(X^{q-4}+X^{q-5}Y+\cdots+Y^{q-4})\alpha_{q-1}(X,Y)]\end{aligned} \tag{17}$$

This implies that either $\alpha_1(X,Y)$ is a multiple of XY or $\alpha_1(X,Y)=0$.

By the GGM assumption, we have that $\alpha_1(X,Y)=0$.

We continue by replacing $\alpha_1(X,Y)=0$ in equation (17):

$$\begin{aligned}0=&\alpha_3(X,Y)+\cdots+(X^{q-4}+X^{q-5}Y+\cdots+Y^{q-4})\alpha_{q-1}(X,Y)\\ -\alpha_3(X,Y)=&(X^2+XY+Y^2)\alpha_4(X,Y)+\cdots+(X^{q-4}+X^{q-5}Y+\cdots+Y^{q-4})\alpha_{q-1}(X,Y)\end{aligned} \tag{18}$$

And so on... till we show that $\alpha_i(X,Y)=0 \ \ \forall i=0\dots q-1$. □

A.2 ASDGP Assumption in GGM

Assumption A.3 (ASDGP)**.** *The q-ASDGP assumption holds for the bilinear group generator $\mathcal{G}$ if for all* PPT *adversaries $\mathcal{A}$ we have, on the probability space* $\mathsf{gk}=(p,\mathbb{G}_1,\mathbb{G}_2,\mathbb{G}_T)\leftarrow\mathcal{G}(1^\lambda)$, $g\leftarrow\$\,\mathbb{G}_1, h\leftarrow\$\,\mathbb{G}_2$ *and* $a,b\leftarrow\$\,\mathbb{Z}_p$ *the following holds:*

$$\Pr\left[\begin{array}{c|r}(\mathbf{A}\neq\mathbf{1}_{\mathbb{G}_1}\vee\ \mathbf{B}\neq\mathbf{1}_{\mathbb{G}_2})&g\leftarrow\$\,\mathbb{G}_1,h\leftarrow\$\,\mathbb{G}_2,a,b\leftarrow\$\,\mathbb{Z}_p\\ \wedge\ \prod_{i=0}^{q-1}e(A_i,h^{a^i})\prod_{i=q}^{2q-1}e(g^{a^i},B_i)=1_{\mathbb{G}_T}&\sigma\leftarrow([g^{a^i},g^{b^i},h^{a^i},h^{b^i}]_{i=0}^{2q-1})\\ \wedge\ \prod_{i=0}^{q-1}e(A_i,h^{b^i})\prod_{i=q}^{2q-1}e(g^{b^i},B_i)=1_{\mathbb{G}_T}&(\mathbf{A},\mathbf{B})\leftarrow\mathcal{A}(\mathsf{gk},\sigma)\end{array}\right]=\mathsf{negl}(\lambda)$$

Lemma A.4. *The q-ASDGP assumption holds in the generic group model.*

Proof. Suppose $\mathcal{A}$ is an adversary that on input (gk,σ), outputs $(A_0,\dots,A_{q-1}),(B_0,\dots,B_{q-1})$ such that $\prod_{i=0}^{q-1}e(A_i,h^{a^i})\prod_{i=q}^{2q-1}e(g^{a^i},B_i)=1_{\mathbb{G}_T}$ and $\prod_{i=0}^{q-1}e(A_i,h^{b^i})\prod_{i=q}^{2q-1}e(g^{b^i},B_i)=1_{\mathbb{G}_T}$. Then its GGM extractor outputs $\alpha_i(X,Y)=\sum_{j=0}^{2q-1}(x_jX^j+y_jY^j+c_j)$ and $\beta_i(X,Y)=\sum_{j=0}^{2q-1}(x_jX^j+y_jY^j+c_j)$ then we have:

$$\alpha_0(X,Y)+X\alpha_1(X,Y)+\cdots+X^{q-1}\alpha_{q-1}(X,Y)+X^q\beta_0(X,Y)+\cdots+X^{2q-1}\beta_{q-1}(X,Y)=0 \tag{19}$$

$$\alpha_0(X,Y)+Y\alpha_1(X,Y)+Y^2\alpha_2(X,Y)+Y^q\beta_0(X,Y)+\cdots+Y^{2q-1}\beta_{q-1}(X,Y)=0 \tag{20}$$

By substracting (20) and (19) we got

$$0 = (X-Y)\alpha_1(X,Y) + \cdots + (X^{q-1} - Y^{q-1})\alpha_{q-1}(X,Y) + (X^q - Y^q)\beta_q(X,Y) + \dots \quad (21)$$

Now we can factor $(X-Y)$ and then divide by it and obtain:

$$\begin{aligned} -\alpha_1(X,Y) =& (X+Y)\alpha_2(X,Y) + (X^2 + XY + Y^2)\alpha_3(X,Y) + \cdots + \\ &+ (X^{2q-2} + YX^{2q-3} + \cdots + Y^{2q-3}X + Y^{2q-2})\beta_{2q-1}(X,Y) \end{aligned} \quad (22)$$

Substitute $-\alpha_1(X,Y)$ in equation (19) and remark that all $X^i\alpha_i(X,Y), X^{q+i}\beta_{q+i}(X,Y)$ terms are vanishing:

$$\begin{aligned} \alpha_0(X,Y) &= X\left[\sum_{i=2}^{q-1}\left(\sum_{j=0}^{i-1} X^{i-j-1}Y^j\right)\alpha_i(X,Y) + \sum_{i=q}^{2q-1}\left(\sum_{j=0}^{i-1} X^{i-j-1}Y^j\right)\beta_i(X,Y)\right] - \\ &\quad - \sum_{i=2}^{q-1} X^i\alpha_i(X,Y) - \sum_{i=q}^{2q-1} X^i\beta_i(X,Y) \\ \alpha_0(X,Y) &= X\left[\sum_{i=2}^{q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^j\right)\alpha_i(X,Y) + \sum_{i=q}^{2q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^j\right)\beta_i(X,Y)\right] \\ \alpha_0(X,Y) &= XY\left[\sum_{i=2}^{q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^{j-1}\right)\alpha_i(X,Y) + \sum_{i=q}^{2q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^{j-1}\right)\beta_i(X,Y)\right] \end{aligned} \quad (23)$$

This implies that either $\alpha_0(X,Y)$ is a multiple of XY or $\alpha_0(X,Y) = 0$.

By the GGM assumption, we have that $\alpha_0(X,Y) = 0$.

We continue by replacing $\alpha_0(X,Y) = 0$ in equation (23):

$$-\alpha_2(X,Y) = \sum_{i=3}^{q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^{j-1}\right)\alpha_i(X,Y) + \sum_{i=q}^{2q-1}\left(\sum_{j=1}^{i-1} X^{i-j-1}Y^{j-1}\right)\beta_i(X,Y) \quad (24)$$

Substitute the expression of $-\alpha_2(X,Y)$ in equation (19) or (20) and remark that all terms $X^i\alpha_i(X,Y), X^i\beta_i(X,Y)$ (respectively $Y^i\alpha_i(X,Y), Y^i\beta_i(X,Y)$) terms are vanishing

And so on... till we show that $\alpha_i(X,Y) = 0 \quad \forall i = 0 \dots q-1$ and $\beta_i(X,Y) = 0 \ \forall i = q \dots 2q-$. □